Network Science (VU) (706.703)

Introduction to Dynamical Systems

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Outline

1. Dynamical Systems
2. Fixed Points
3. Geometry of fixed points
4. Logistic Growth
5. Linear Stability Analysis
6. Linear systems with two variables
7. Linear stability analysis for multi-variable systems
8. Numerical Solutions

Definition of dynamical systems

- We now first focus now on dynamical systems in a non-network context
- We also concentrate on the deterministic systems of continous real-valued variables evolving in continous time t
- A simple example is a system described by a single variable $x(t)$
- The variable evolves according to a first-order differential equation:

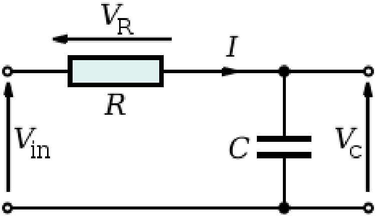
$$\frac{dx}{dt} = \dot{x} = f(x)$$

- Henceforth, we will denote the time derivative of x with $\dot{x}$

Definition of dynamical systems

- $f(x)$ is some specified function that describes the behavior of x
- Typically we also have initial conditions (for an initial value problem)
- The value $x(t_0)$ at some initial time t_0
- For example, the RC circuit from the electrical engineering

Example: RC Circuit



Example: RC Circuit

- Let us write the equations (Kirchhoff's voltage law)
- As we go around the circuit the sum of voltage equals zero:

$$-V_{in} + RI + \frac{Q}{C} = 0$$

Example: RC Circuit

- The change of electrical charge in time is the electrical current:

$$\dot{Q} = I$$

$$\dot{Q} = f(Q) = \frac{V_{in}}{R} - \frac{Q}{RC}$$

- And we might have an initial condition: $Q(0) = 0$ (capacitor is empty in the beginning)

Definition of dynamical systems

- We can have dynamical systems with two variables:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{aligned}
$$

- We can extend this approach to even more variables

General framework

- A dynamical system with n variables:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_n) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n)
\end{aligned}$$

General framework

- We might have also the right side dependence on t, e.g:

$$\dot{x}_1 = f_1(x_1, t)$$

- However, we can easily rewrite this equation in one without dependence on t, but with one extra variable

General framework

$$x_2 = t \implies \dot{x}_2 = 1$$

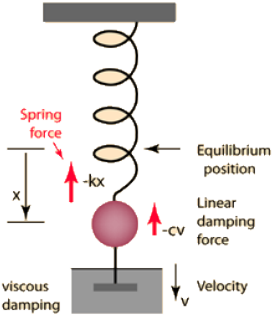
- And we also have: $x_2(0) = 0$

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) = 1\end{aligned}$$

General framework

- Another extension would be to consider systems governed by higher derivatives
- It turns out that these can always be reduced to simpler cases
- However, we need to introduce extra variables
- For example, damped harmonic oscillator

Example: Damped harmonic oscillator



Example: Damped harmonic oscillator

- Let us write the equations (Newton's second law of motion)
- $\vec{F} = m\vec{a}$
- $a = \ddot{x} = \frac{d^2x}{dt^2}$
- $v = \dot{x}$

$$m\ddot{x} = -c\dot{x} - kx$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

Example: Damped harmonic oscillator

- Now let us define: $x_1 = x$ and $x_2 = \dot{x}$
- This implies $\dot{x}_1 = x_2$

$$m\dot{x}_2 + cx_2 + kx_1 = 0$$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2\end{aligned}$$

General framework

- The examples were all examples of **linear** systems because all of the x_i on the right hand side are to the first power only
- Otherwise the systems are **nonlinear**
- Nonlinear terms are products, powers, e.g. $x_1 x_2$, x_1^2, and so on
- Further nonlinear terms are (nonlinear) functions of x_i, e.g. $sin x_i$, or $log x_i$, and so on
- With nonlinearity the study of even such simple dynamical systems covers a broad range of interesting scientific situations

Exponential growth/decay equation

- Linear systems with a single variable exhibit exponential growth/decay behavior
- For example exponential growth equation

$$\dot{x} = kx$$

- Where $k > 0$ is the growth rate
- We might have the following initial condition: $x(0) = x_0$

Exponential growth/decay equation

- Such simple systems can be solved analytically by separating variables and integrating

$$\begin{aligned}\frac{dx}{dt} &= kx \\ \frac{dx}{x} &= kdt \\ \int \frac{dx}{x} &= \int kdt\end{aligned}$$

Exponential growth/decay equation

- Solving integrals:

$$
\begin{aligned}
ln x &= kt + c \\
x &= e^{kt}e^{c} = Ce^{kt}
\end{aligned}
$$

Exponential growth/decay equation

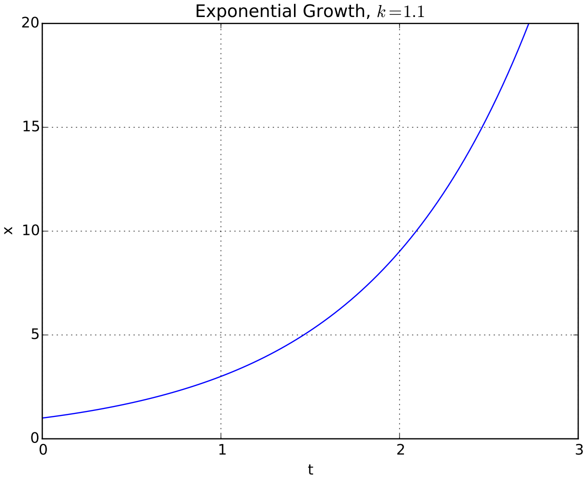
- The constant C is calculated from the initial conditions
- For $t = 0$ we have $x(0) = x_0$

$$\begin{aligned} x_0 &= Ce^{k \cdot 0} = C \cdot 1 \\ C &= x_0 \end{aligned}$$

- The final solution

$$x = x_0 e^{kt}$$

Exponential growth/decay equation



Exponential growth/decay equation

- Similarly exponential decay equation

$$\dot{x} = -\lambda x$$

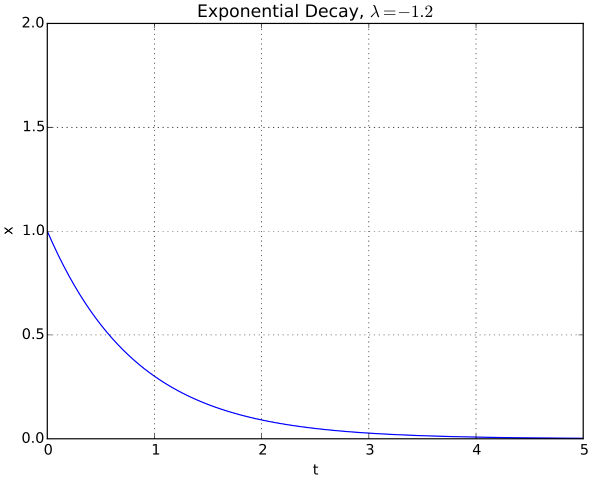
- Where $\lambda > 0$ is the decay rate
- We might have the following initial condition: $x(0) = x_0$

Exponential growth/decay equation

- Again, by separating variables, integrating and calculating integration constants from the initial conditions
- The final solution:

$$x = x_0 e^{-\lambda t}$$

Exponential growth/decay equation



Problems with analytical solutions

- In principle, we can always solve the equation from above by separating the variables and integrating:

$$
\begin{aligned}
\frac{dx}{dt} &= f(x) \\
\int_{x_0}^{x} \frac{dx'}{f(x')} &= t - t_0
\end{aligned}
$$

Problems with analytical solutions

- In practice, the integral may not exist in the closed form
- For cases with two or more variables it is not even in principle possible to find solution in a general case
- We will see later that for the network cases we typically have n variables: one variable per node
- Thus, except in some special cases a full analytical solution is typically not possible
- We can of course always integrate equations numerically or simulate
- But, combining these methods with some geometric and analytical techniques provides us with more qualitative insight

Fixed points

- A fixed point is a steady state of the system
- Any value of the variable(s) for which the system is stationary
- The system does not change over time
- Equilibrium

Fixed points

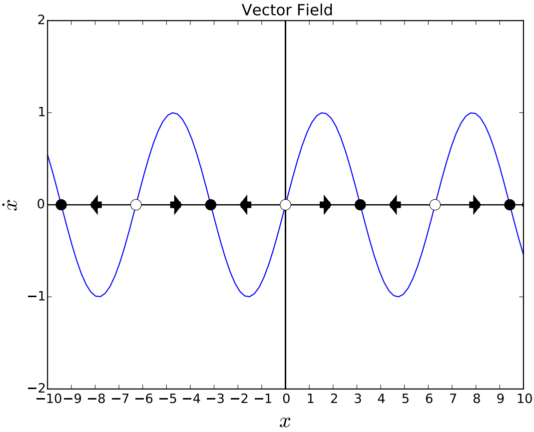
- For example in a system with one variable x a fixed point x^* is any point for which the function $f(x)$ does not change:

$$f(x^*) = 0$$

- This makes $\frac{dx}{dt} = 0$, and x does not move
- Thus, if in the evolution of the system we reach a fixed point the system stays there forever

Vector field

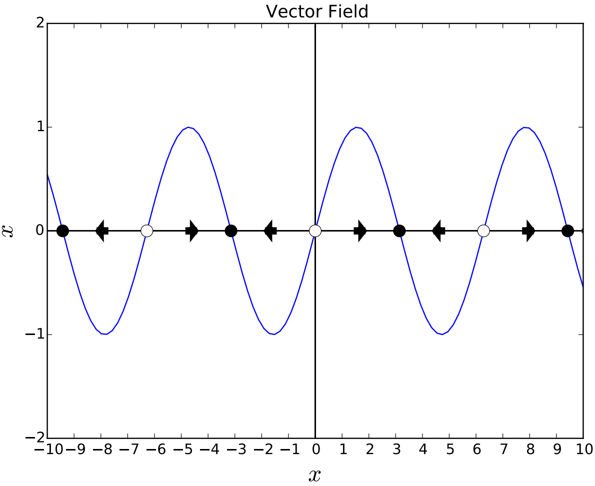
- We plot $\dot{x}$ vs x, e.g. $\dot{x} = sinx$



Vector field

- The arrows are the vector field
- Imagine that an object (e.g.) a car is moving along the x-axis
- x is its position in time
- $\dot{x}$ is then its velocity
- The velocity varies from place to place according to $\dot{x} = sinx$

Vector field



Vector field

- Where is the object moving when $\dot{x} > 0$

Vector field

- Where is the object moving when $\dot{x} > 0$
- To the right
- Where is the object moving when $\dot{x} < 0$

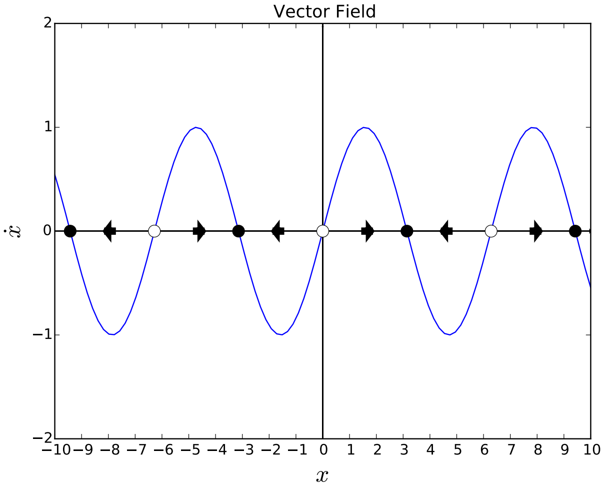
Vector field

- Where is the object moving when $\dot{x} > 0$
- To the right
- Where is the object moving when $\dot{x} < 0$
- To the left
- Where is the object moving when $\dot{x} = 0$

Vector field

- Where is the object moving when $\dot{x} > 0$
- To the right
- Where is the object moving when $\dot{x} < 0$
- To the left
- Where is the object moving when $\dot{x} = 0$
- Nowhere: it stays in the same place
- These are the fixed points

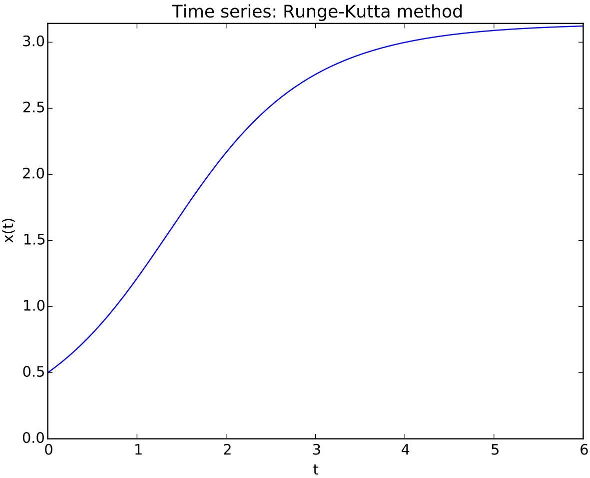
Vector field



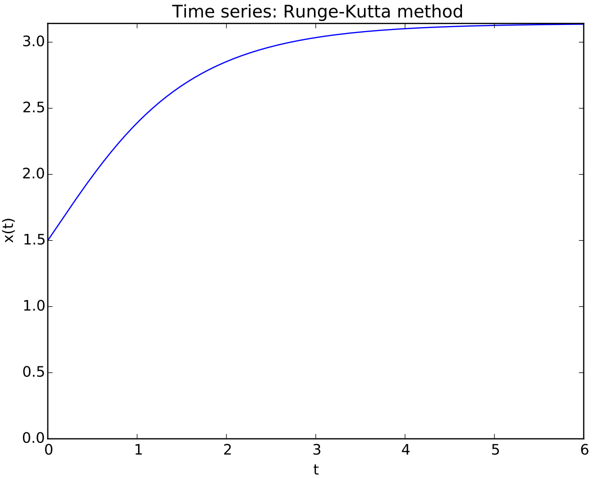
Vector field

- Two kinds of the fixed points
- What will happen if we are at a fixed point, e.g. $x = \pi$ and move slightly left or right
- We are attracted back to those fixed points: these are the **stable** fixed points
- What will happen if we are at a fixed point, e.g. $x = 2\pi$ and move slightly left or right
- We are repelled away from those fixed points: these are the **unstable** fixed points

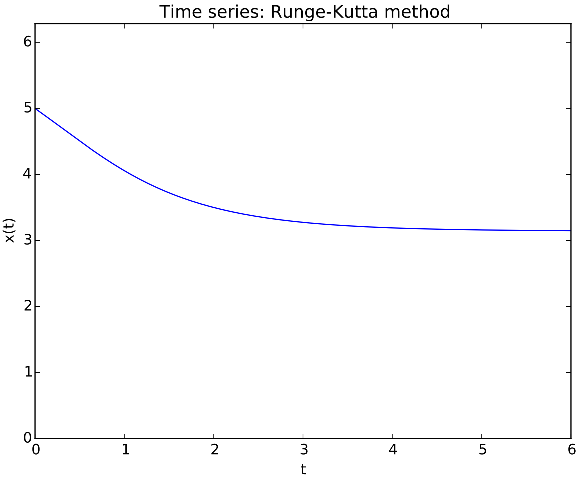
Time series: trajectories



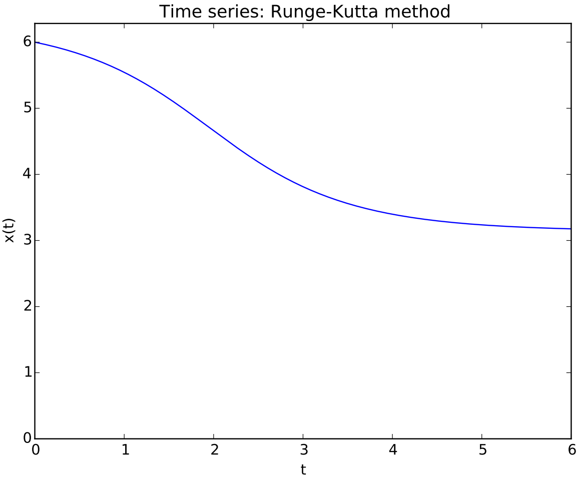
Time series: trajectories



Time series: trajectories



Time series: trajectories



Vector field: examples

- Find all fixed points and classify their stability:
  1. $\dot{x} = x^2 - 1$
  2. $\dot{x} = x - x^3$
  3. RC circuit: $\dot{Q} = \frac{V_{in}}{R} - \frac{Q}{RC}$
  4. $\dot{x} = x - cosx$
  5. $\dot{x} = e^x - cosx$

Logistic growth equation

- The simplest population growth model is the exponential growth model: $\dot{N} = rN$, with $r > 0$ being the growth rate
- This model predicts the exponential growth: $N = N_0 e^{rt}$, where N_0 is the population at time $t = 0$
- Of course, such exponential growth can not go forever
- For population larger then some (positive) carrying capacity K the growth rate becomes actually negative
- The death rate is higher than the birth rate

Logistic growth equation

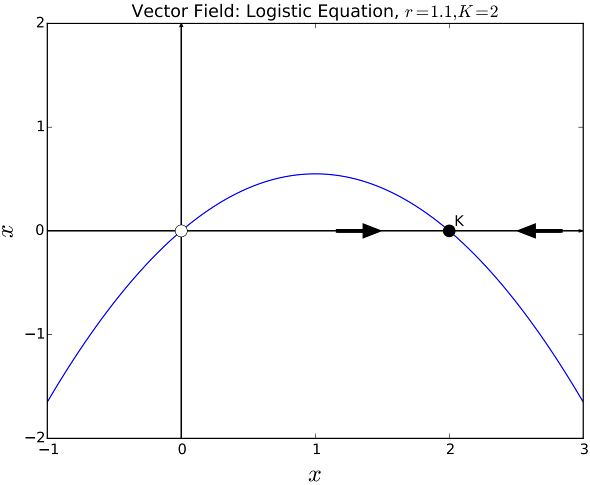
- To model the effects of overcrowding and limited resources we will assume that per capita growth rate $\frac{\dot{N}}{N}$ decreases when N is sufficiently large
- A mathematically convenient solution is to assume that per capita growth rate $\frac{\dot{N}}{N}$ decreases linearly with N

$$\frac{\dot{N}}{N} = r(1 - \frac{N}{K})$$

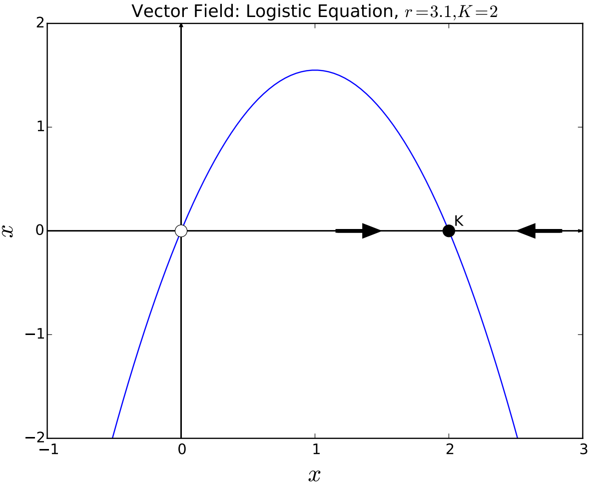
$$\dot{N} = rN(1 - \frac{N}{K})$$

- This is the *logistic growth equation*

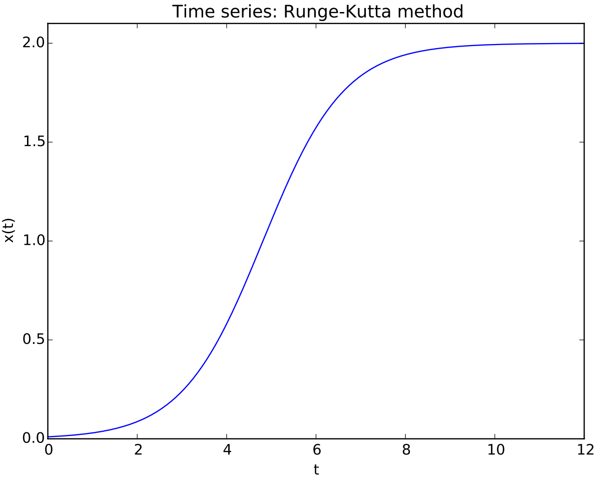
Vector field: logistic equation



Vector field: logistic equation



Time series: logistic equation



Logistic growth equation

- Logistic growth equation can be solved analytically by separating variables

$$
\begin{aligned}
\frac{dN}{(1-\frac{N}{K})N} &= rdt \\
\int \frac{dN}{(1-\frac{N}{K})N} &= \int rdt
\end{aligned}
$$

Logistic growth equation

- For the integral on the left side we use partial fractions expansion:

$$\frac{1}{(1-\frac{N}{K})N} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}}$$

$$\begin{aligned}
A - A\frac{N}{K} + BN &= 1 \\
A + N(B - \frac{A}{K}) &= 1 \\
\Longrightarrow A &= 1 \\
\Longrightarrow \quad B - \frac{1}{K} &= 0 \\
B &= \frac{1}{K}
\end{aligned}$$

Logistic growth equation

- For the integral on the left side we use partial fractions expansion:

$$\frac{1}{(1-\frac{N}{K})N} = \frac{1}{N} + \frac{1}{K-N}$$

$$\begin{aligned}\int \frac{dN}{(1-\frac{N}{K})N} &= \int \frac{dN}{N} + \int \frac{dN}{K-N} = lnN - ln(K-N) \\ &= ln\frac{N}{K-N}\end{aligned}$$

Logistic growth equation

- For the right side we have:

$$\int rdt = rt + c$$

- Thus, we obtain (with $C = e^c$):

$$\begin{aligned} ln\frac{N}{K-N} &= rt + c \\ \frac{N}{K-N} &= e^{rt}e^{c} = Ce^{rt} \end{aligned}$$

Logistic growth equation

- Now we solve for N

$$
\begin{aligned}
\frac{N}{K-N} &= Ce^{rt} \\
N &= CKe^{rt} - CNe^{rt} \\
N(1 + Ce^{rt}) &= CKe^{rt} \\
N &= \frac{CKe^{rt}}{1 + Ce^{rt}}
\end{aligned}
$$

Logistic growth equation

- The constant C is calculated from the initial conditions
- For $t = 0$ we have the initial population N_0

$$\frac{N_0}{K - N_0} = Ce^{r \cdot 0} = C \cdot 1$$
$$C = \frac{N_0}{K - N_0}$$

Logistic growth equation

- By substituting $C = \frac{N_0}{K-N_0}$ and simplifying:

$$x = \frac{KN_0e^{rt}}{K - N_0 + N_0e^{rt}}$$

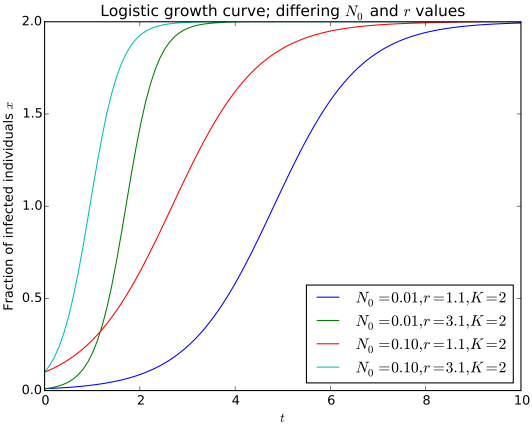
- By dividing with e^{rt}, rearranging, and dividing with K:

$$x = \frac{KN_0}{N_0 + e^{-rt}(K - N_0)}$$

$$x = \frac{N_0}{\frac{N_0}{K} + e^{-rt}(1 - \frac{N_0}{K})}$$

- This is *logistic growth curve*

Logistic growth curve



Python Notebook

- Check logistic growth examples from python notebook
- http://kti.tugraz.at/staff/denis/courses/netsci/dynamics.ipynb

Fixed points

- Recollect: In a system with one variable x a fixed point x^* is any point for which the function $f(x)$ does not change:

$$f(x^*) = 0$$

- This makes $\frac{dx}{dt} = 0$, and x does not move
- Thus, if in the evolution of the system we reach a fixed point the system stays there forever

Fixed points

- In a two variable system, a fixed point is a pair of values such that:

$$
\begin{aligned}
f(x^*, y^*) &= 0 \\
g(x^*, y^*) &= 0
\end{aligned}
$$

- This makes $\frac{dx}{dt} = \frac{dy}{dt} = 0$, and x and y do not move

Fixed points: example

- The logistic model: $\dot{N} = f(N) = rN(1 - \frac{N}{K})$

$$
\begin{aligned}
rN(1 - \frac{N}{K}) &= 0 \\
N &= 0 \\
N &= K
\end{aligned}
$$

- $N = 0$ there is no one in the population and no reproduction is possible
- $N = K$ the population size reached its limit

Linearization

- It is easy to find fixed points
- It is straightforward to analyze the dynamics of the system in the vicinity of the fixed points
- Let us take a look at one-variable system
- We represent the value of x close to x^* by: $x = x^* + \epsilon$, for some small ϵ:

$$\frac{dx}{dt} = \frac{d\epsilon}{dt} = f(x^* + \epsilon)$$

Linearization

- Taylor expansion of the right-hand side about the point $x = x^*$:

$$\frac{d\epsilon}{dt} = f(x^*) + \epsilon f'(x^*) + O(\epsilon^2)$$

- f' is the derivative of f with respect to its arguments

Linearization

- Neglecting terms of order $O(\epsilon^2)$ (because ϵ is small)
- Also, $f(x^*) = 0$

$$\frac{d\epsilon}{dt} = \epsilon f'(x^*)$$

Linearization

- Linear first-order differential equation which can be solved by separating variables:

$$
\begin{aligned}
\epsilon(t) &= \epsilon(0)e^{\lambda t} \\
\lambda &= f'(x^*)
\end{aligned}
$$

- λ is just a number, which we calculate by evaluating f' at fixed point x^*

Linearization

- Depending on the sign of λ we may have attracting fixed and repelling fixed points
- E.g. if $\lambda < 0$ points close to the fixed point are attracted to it
- If $\lambda > 0$ points close to the fixed point are repelled away
- If $\lambda = 0$ points close to the fixed point are neither attracted nor repelled
- This kind of analysis is called linear stability analysis

Fixed points: Logistic model

$$\begin{aligned} f(N) &= rN(1-\frac{N}{K}) = rN - \frac{r}{K}N^2 \\ f'(N) &= r - 2\frac{r}{K}N \end{aligned}$$

$$\begin{aligned} N_0^* &= 0 \\ f'(N_0^*) &= r \\ N_1^* &= K \\ f'(N_1^*) &= -r \end{aligned}$$

- $N^* = 0$, repelling fixed point (exponential growth in the beginning)
- $N^* = 1$, attracting fixed point (saturation in the end)

Two-dimensional linear system

- A two-dimensional linear system is of the form:

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2 \end{aligned}$$

- a, b, c, d are parameters

Two-dimensional linear system

- In matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Two-dimensional linear system

- The system is linear also in another sense
- If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions so is any linear combination: $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$
- $\dot{\mathbf{x}} = 0$ when $\mathbf{x} = 0$
- $\mathbf{x}^* = 0$ is always a fixed point for any choice of $\mathbf{A}$

Solutions for two-dimensional linear systems

- Generalizing from the one-dimensional linear system, the solutions for a two-dimensional linear systems will be of the form:

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

- This corresponds to an exponential growth/decay alongside the line spanned by the vector $\mathbf{v}$

Solutions for two-dimensional linear systems

- Let us find the solutions
- We substitute $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$$\lambda e^{\lambda t}\mathbf{v} = \mathbf{A}e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v}$$

- Canceling $e^{\lambda t}$ we get:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Solutions for two-dimensional linear systems

- The straight line solutions are eigenvectors of $\mathbf{A}$
- The growth rate/decay is given by the eigenvalues of $\mathbf{A}$
- If the corresponding eigenvalue is smaller than zero we have an exponential decay alongside that eigenvector
- If the corresponding eigenvalue is greater than zero we have an exponential growth alongside that eigenvector
- Larger eigenvalue is a fast eigendirection, smaller eigenvalue is a slow eigendirection
- These are **eigensolutions**

Solutions for two-dimensional linear systems

- If $\lambda_1 \neq \lambda_2$ the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent
- Then any initial condition $\mathbf{x}_0$ can be written as linear combination of eigenvectors:

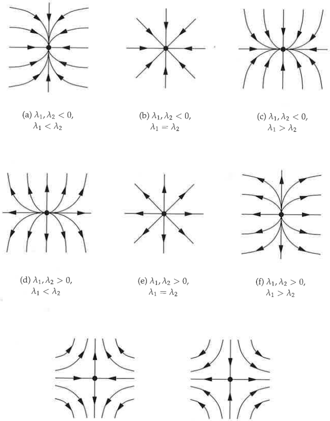
$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

- The general solution for $\mathbf{x}(t)$:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

- It is a linear combination of solutions to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, i.e. it is itself a solution
- It satisfies the initial conditions: it is the only solution

Flows in two-dimensional linear systems



(a) $\lambda_1, \lambda_2 < 0$, $\lambda_1 < \lambda_2$

(b) $\lambda_1, \lambda_2 < 0$, $\lambda_1 = \lambda_2$

(c) $\lambda_1, \lambda_2 < 0$, $\lambda_1 > \lambda_2$

(d) $\lambda_1, \lambda_2 > 0$, $\lambda_1 < \lambda_2$

(e) $\lambda_1, \lambda_2 > 0$, $\lambda_1 = \lambda_2$

(f) $\lambda_1, \lambda_2 > 0$, $\lambda_1 > \lambda_2$

(g) $\lambda_1 < 0 < \lambda_2$

(h) $\lambda_2 < 0 < \lambda_1$

Flows in two-dimensional linear systems

- If $\mathbf{A}$ is not symmetric eigenvectors are not orthogonal
- This transforms the axes, but the behavior is similar
- A new interesting behavior might emerge if the eigenvalues are complex
- This gives an oscillation around a fixed point, which either grows or decays
- It spirals inwards or outwards around the fixed point
- In certain cases there is a stable oscillatory behavior: limit cycle

Example: Romeo and Juliet

- Romeo and Juliet are in a love affair (Strogatz 1998)
- Let us define:

$$
\begin{aligned}
R(t) &= \text{Romeo's love/hate for Juliet in time } t \\
J(t) &= \text{Juliet's love/hate for Romeo in time } t
\end{aligned}
$$

- Positive values of R and J signify love, negative hate.

Example 1: Romeo and Juliet

- Romeo and Juliet love only themselves:

$$
\begin{aligned}
\dot{R} &= aR \\
\dot{J} &= bJ
\end{aligned}
$$

$$
\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}
$$

- a and b are positive
- The initial conditions: $\mathbf{x}(0) = \begin{pmatrix} R_0 \\ J_0 \end{pmatrix}$

Example 1: Romeo and Juliet

- The eigenvalues of a diagonal matrix are on the diagonal: $\lambda_1 = a$, $\lambda_2 = b$
- The eigenvectors of a diagonal matrix form the basis of the Euclidean space: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- The solution is of the form:

$$\mathbf{x}(t) = c_1 e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{bt} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example 1: Romeo and Juliet

- From initial conditions:

$$\mathbf{x}(0) = \begin{pmatrix} R_0 \\ J_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{x}(0) = R_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + J_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example 1: Romeo and Juliet

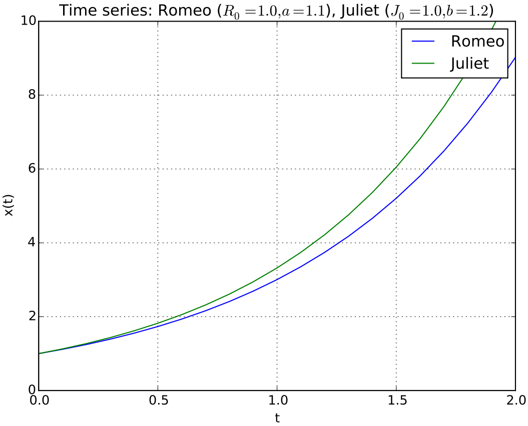
- The final solution:

$$\mathbf{x}(t) = R_0 e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + J_0 e^{bt} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

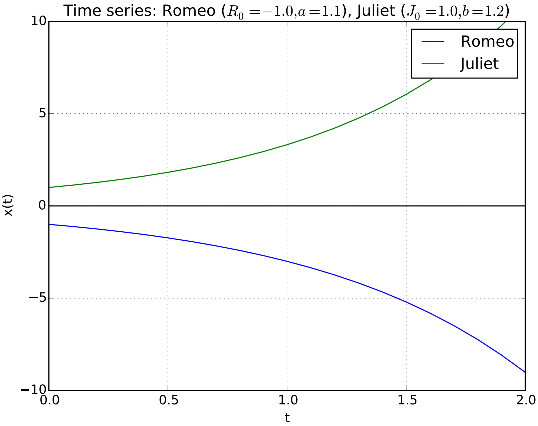
$$\begin{aligned} R(t) &= R_0 e^{at} \\ J(t) &= J_0 e^{bt} \end{aligned}$$

- They evolve independently (Romeo and Juliet are decoupled)

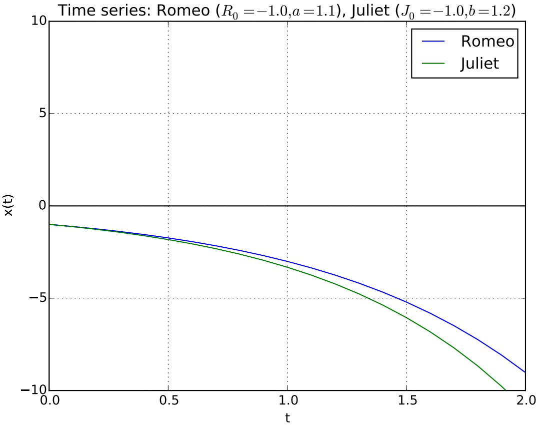
Example 1: Romeo and Juliet



Example 1: Romeo and Juliet



Example 1: Romeo and Juliet



Example 2: Romeo and Juliet

- Romeo and Juliet react only to each other, but not to themselves:

$$\begin{aligned} \dot{R} &= aJ \\ \dot{J} &= bR \end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

- a and b are positive
- The initial conditions: $\mathbf{x}(0) = \begin{pmatrix} R_0 \\ J_0 \end{pmatrix}$

Example 2: Romeo and Juliet

- The eigenvalues: $\lambda_1 = \sqrt{ab}$, $\lambda_2 = -\sqrt{ab}$
- The eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} \sqrt{\frac{a}{b}} \\ 1 \end{pmatrix}$ $\mathbf{v}_2 = \begin{pmatrix} \sqrt{\frac{a}{b}} \\ -1 \end{pmatrix}$
- The solution is of the form:

$$\mathbf{x}(t) = c_1 e^{(\sqrt{ab})t} \begin{pmatrix} \sqrt{\frac{a}{b}} \\ 1 \end{pmatrix} + c_2 e^{(\sqrt{ab})t} \begin{pmatrix} \sqrt{\frac{a}{b}} \\ -1 \end{pmatrix}$$

Example 2: Romeo and Juliet

- From initial conditions:

$$\mathbf{x}(0) = \begin{pmatrix} R_0 \\ J_0 \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{\frac{a}{b}} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{\frac{a}{b}} \\ -1 \end{pmatrix}$$

$$\mathbf{x}(0) = \frac{1}{2}(\sqrt{\frac{b}{a}}R_0 + J_0) \begin{pmatrix} \sqrt{\frac{a}{b}} \\ 1 \end{pmatrix} + \frac{1}{2}(\sqrt{\frac{b}{a}}R_0 - J_0) \begin{pmatrix} \sqrt{\frac{a}{b}} \\ -1 \end{pmatrix}$$

Example 2: Romeo and Juliet

- The final solution:

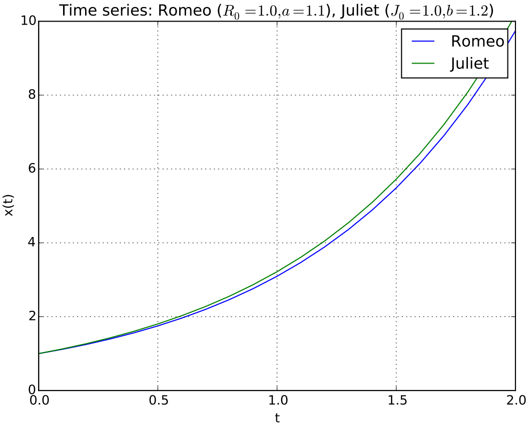
$$\mathbf{x}(t) = \frac{1}{2}(\sqrt{\frac{b}{a}}R_0 + J_0)e^{(\sqrt{ab})t}\begin{pmatrix}\sqrt{\frac{a}{b}} \\ 1\end{pmatrix} + \frac{1}{2}(\sqrt{\frac{b}{a}}R_0 - J_0)e^{(-\sqrt{ab})t}\begin{pmatrix}\sqrt{\frac{a}{b}} \\ -1\end{pmatrix}$$

$$\begin{aligned} R(t) &\approx \frac{1}{2}(R_0 + \sqrt{\frac{a}{b}}J_0)e^{(\sqrt{ab})t} \\ J(t) &\approx \frac{1}{2}(\sqrt{\frac{b}{a}}R_0 + J_0)e^{(\sqrt{ab})t} \end{aligned}$$

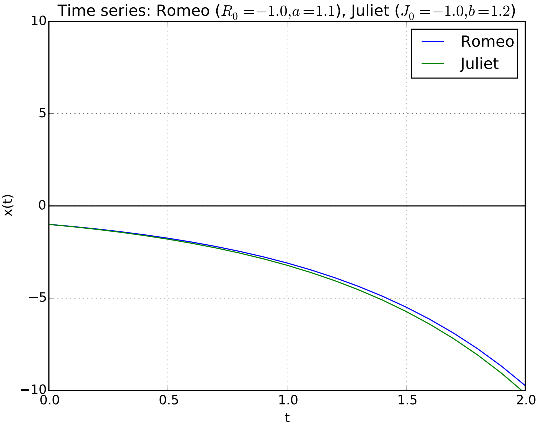
Example 2: Romeo and Juliet

- Possibilities:
  1. $R_0 > 0$, $J_0 > 0$ then the dynamics evolves into a love fest
  2. $R_0 < 0$, $J_0 < 0$ then the dynamics evolves into a war
  3. $R_0 + \sqrt{\frac{a}{b}} J_0 > 0$ then the dynamics evolves into a love fest
  4. $R_0 + \sqrt{\frac{a}{b}} J_0 < 0$ then the dynamics evolves into a war

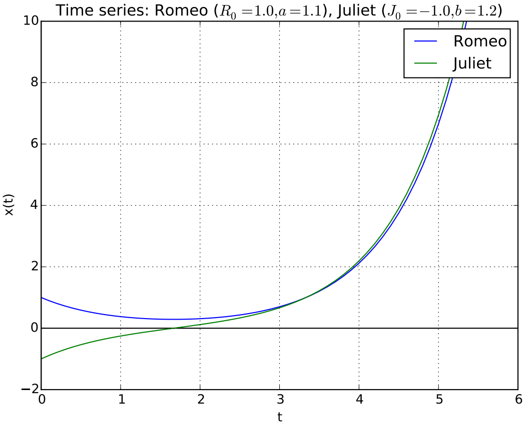
Example 2: Romeo and Juliet



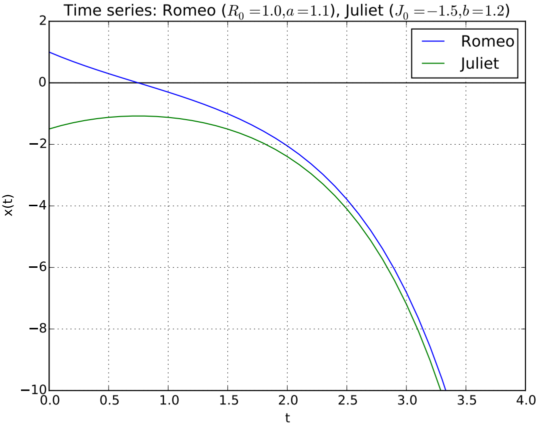
Example 2: Romeo and Juliet



Example 2: Romeo and Juliet



Example 2: Romeo and Juliet



Example 3: Romeo and Juliet

- The more Romeo loves Juliet, the more Juliet wants to run away and hide:

$$\begin{aligned} \dot{R} &= aJ \\ \dot{J} &= -bR \end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$$

- a and b are positive
- The initial conditions: $\mathbf{x}(0) = \begin{pmatrix} R_0 \\ J_0 \end{pmatrix}$

Example 3: Romeo and Juliet

- The eigenvalues: $\lambda_1 = i\sqrt{ab}$, $\lambda_2 = -i\sqrt{ab}$
- The eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} i\sqrt{\frac{a}{b}} \\ -1 \end{pmatrix}$ $\mathbf{v}_2 = \begin{pmatrix} i\sqrt{\frac{a}{b}} \\ 1 \end{pmatrix}$
- The solution is of the form:

$$\mathbf{x}(t) = c_1 e^{(i\sqrt{ab})t} \begin{pmatrix} i\sqrt{\frac{a}{b}} \\ -1 \end{pmatrix} + c_2 e^{(-i\sqrt{ab})t} \begin{pmatrix} i\sqrt{\frac{a}{b}} \\ 1 \end{pmatrix}$$

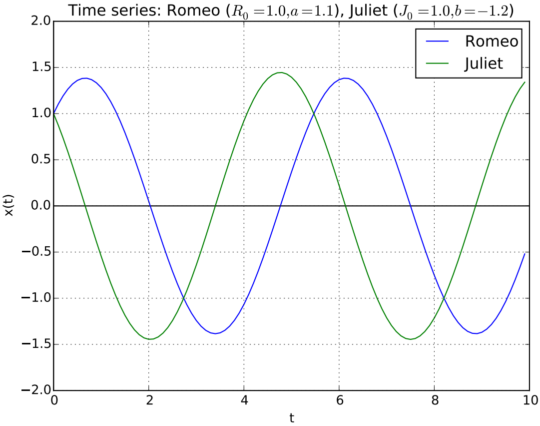
Example 3: Romeo and Juliet

- The final solution can be obtained by using Euler formula $(e^{ix} = cosx + isinx)$:

$$
\begin{aligned}
R(t) &= R_0 cos((\sqrt{ab})t) + J_0\sqrt{\frac{a}{b}} sin((\sqrt{ab})t) \\
J(t) &= J_0 cos((\sqrt{ab})t) - R_0\sqrt{\frac{b}{a}} sin((\sqrt{ab})t)
\end{aligned}
$$

- Never-ending cycle of love and hate

Example 3: Romeo and Juliet



Python Notebook

- Check Romeo and Juliet examples from python notebook
- http://kti.tugraz.at/staff/denis/courses/netsci/dynamics.ipynb

Linearization with two variables

- For a fixed point x^* and y^*:

$$
\begin{aligned}
f(x^*, y^*) &= 0 \\
g(x^*, y^*) &= 0
\end{aligned}
$$

- We represent points close to the fixed point as $x = x^* + \epsilon_x$ and $y = y^* + \epsilon_y$:

Linearization with two variables

- As before we expand about the fixed point, performing a double Taylor expansion

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d\epsilon_x}{dt} = f(x^* + \epsilon_x, y^* + \epsilon_y) \\
&= f(x^*, y^*) + \epsilon_x \left(\frac{\partial f}{\partial x}\right)_{\substack{x=x^* \\ y=y^*}} + \epsilon_y \left(\frac{\partial f}{\partial y}\right)_{\substack{x=x^* \\ y=y^*}} + O(\epsilon_x^2) + O(\epsilon_y^2)
\end{aligned}$$

Linearization with two variables

- Ignoring all higher-order terms in the expnasion:

$$\begin{aligned}
\frac{d\epsilon_x}{dt} &= \epsilon_x \left(\frac{\partial f}{\partial x}\right)_{\substack{x=x^* \\ y=y^*}} + \epsilon_y \left(\frac{\partial f}{\partial y}\right)_{\substack{x=x^* \\ y=y^*}} \\
\frac{d\epsilon_y}{dt} &= \epsilon_x \left(\frac{\partial g}{\partial x}\right)_{\substack{x=x^* \\ y=y^*}} + \epsilon_y \left(\frac{\partial g}{\partial y}\right)_{\substack{x=x^* \\ y=y^*}}
\end{aligned}$$

Linearization with two variables

- In the matrix form:

$$\frac{d\boldsymbol{\epsilon}}{dt} = \mathbf{J}\boldsymbol{\epsilon}$$

- $\boldsymbol{\epsilon}$ is the vector with $\begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix}$ and $\mathbf{J}$ is the Jacobian matrix evaluated at the fixed point:

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

Linearization with two variables

- Now we have a linear system and solve it in the usual way
- However, with the Linearization we are able only to analyze the system nearby fixed points
- Further away the behavior may change
- For systems with three and more variables we apply the same approach
- The rank of vectors and matrices increases with the increasing number of variables

Linearization: example

- Find all the fixed points of the system:

$$
\begin{aligned}
\dot{x} &= -x + x^3 \\
\dot{y} &= -2y
\end{aligned}
$$

- Use linearization to classify their stability

Solving equations on the computer

- We start with the definition of derivative:

$$\dot{x} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

- Now if Δt is sufficiently small we can approximate $\dot{x}$ with $\frac{\Delta x}{\Delta t}$

Solving equations on the computer

- We iterate over time:

$$\begin{aligned} \frac{\Delta x}{\Delta t} &= f(x_n) \\ \frac{x_{n+1} - x_n}{\Delta t} &= f(x_n) \\ x_{n+1} &= x_n + f(x_n)\Delta t \end{aligned}$$

- This is Euler method with local error $O(\Delta t^2)$ and global error $O(\Delta t)$

Solving equations on the computer

- Because of the error you want to choose small Δt
- Also: danger of numerical instability if Δt is not small enough
- But you don't want to choose too small Δt
- Numerical imprecision
- Too many iterations

Solving equations on the computer

- Better solution: alternate time stepping schemes
- Averaging derivative over Δt
- With Euler method we approximate with the derivative at the beginning of the interval
- E.g. improved Euler method:

$$
\begin{aligned}
\hat{x}_{n+1} &= x_n + f(x_n)\Delta t \\
x_{n+1} &= x_n + \frac{1}{2}[f(x_n) + f(\hat{x}_{n+1})]\Delta t
\end{aligned}
$$

- Global error: $O(\Delta t^2)$ but more calculations

Solving equations on the computer

- Runge-Kutta method:

$$
\begin{aligned}
k_1 &= f(x_n)\Delta t \\
k_2 &= f(x_n + \frac{1}{2}k_1)\Delta t \\
k_3 &= f(x_n + \frac{1}{2}k_2)\Delta t \\
k_4 &= f(x_n + k_3)\Delta t \\
x_{n+1} &= x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

- Global error: $O(\Delta t^4)$ but even more calculations