# Network Science (VU) (706.703)

## Empirical Analysis of Networks

Denis Helic

ISDS, TU Graz

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# Outline

# Basic Statistics

| | Network | Type | $n$ | $m$ | $c$ | $S$ | $\ell$ | $\alpha$ | $C$ | $C_{WS}$ | $r$ | Ref(s). |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Social | Film actors | Undirected | 449 913 | 25 516 482 | 113.43 | 0.980 | 3.48 | 2.3 | 0.20 | 0.78 | 0.208 | 16, 323 |
| | Company directors | Undirected | 7 673 | 55 392 | 14.44 | 0.876 | 4.60 | – | 0.59 | 0.88 | 0.276 | 88, 253 |
| | Math coauthorship | Undirected | 253 339 | 496 489 | 3.92 | 0.822 | 7.57 | – | 0.15 | 0.34 | 0.120 | 89, 146 |
| | Physics coauthorship | Undirected | 52 909 | 245 300 | 9.27 | 0.838 | 6.19 | – | 0.45 | 0.56 | 0.363 | 234, 236 |
| | Biology coauthorship | Undirected | 1 520 251 | 11 803 064 | 15.53 | 0.918 | 4.92 | – | 0.088 | 0.60 | 0.127 | 234, 236 |
| | Telephone call graph | Undirected | 47 000 000 | 80 000 000 | 3.16 | | | 2.1 | | | | 9, 10 |
| | Email messages | Directed | 59 812 | 86 300 | 1.44 | 0.952 | 4.95 | 1.5/2.0 | | 0.16 | | 103 |
| | Email address books | Directed | 16 881 | 57 029 | 3.38 | 0.590 | 5.22 | – | 0.17 | 0.13 | 0.092 | 248 |
| | Student dating | Undirected | 573 | 477 | 1.66 | 0.503 | 16.01 | – | 0.005 | 0.001 | −0.029 | 34 |
| | Sexual contacts | Undirected | 2 810 | | | | | 3.2 | | | | 197, 198 |
| Information | WWW nd.edu | Directed | 269 504 | 1 497 135 | 5.55 | 1.000 | 11.27 | 2.1/2.4 | 0.11 | 0.29 | −0.067 | 13, 28 |
| | WWW AltaVista | Directed | 203 549 046 | 1 466 000 000 | 7.20 | 0.914 | 16.18 | 2.1/2.7 | | | | 56 |
| | Citation network | Directed | 783 339 | 6 716 198 | 8.57 | | | 3.0/– | | | | 280 |
| | Roget's Thesaurus | Directed | 1 022 | 5 103 | 4.99 | 0.977 | 4.87 | – | 0.13 | 0.15 | 0.157 | 184 |
| | Word co-occurrence | Undirected | 460 902 | 16 100 000 | 66.96 | 1.000 | | 2.7 | | 0.44 | | 97, 116 |
| Technological | Internet | Undirected | 10 697 | 31 992 | 5.98 | 1.000 | 3.31 | 2.5 | 0.035 | 0.39 | −0.189 | 66, 111 |
| | Power grid | Undirected | 4 941 | 6 594 | 2.67 | 1.000 | 18.99 | – | 0.10 | 0.080 | −0.003 | 323 |
| | Train routes | Undirected | 587 | 19 603 | 66.79 | 1.000 | 2.16 | – | | 0.69 | −0.033 | 294 |
| | Software packages | Directed | 1 439 | 1 723 | 1.20 | 0.998 | 2.42 | 1.6/1.4 | 0.070 | 0.082 | −0.016 | 239 |
| | Software classes | Directed | 1 376 | 2 213 | 1.61 | 1.000 | 5.40 | – | 0.033 | 0.012 | −0.119 | 315 |
| | Electronic circuits | Undirected | 24 097 | 53 248 | 4.34 | 1.000 | 11.05 | 3.0 | 0.010 | 0.030 | −0.154 | 115 |
| | Peer-to-peer network | Undirected | 880 | 1 296 | 1.47 | 0.805 | 4.28 | 2.1 | 0.012 | 0.011 | −0.366 | 6, 282 |
| Biological | Metabolic network | Undirected | 765 | 3 686 | 9.64 | 0.996 | 2.56 | 2.2 | 0.090 | 0.67 | −0.240 | 166 |
| | Protein interactions | Undirected | 2 115 | 2 240 | 2.12 | 0.689 | 6.80 | 2.4 | 0.072 | 0.071 | −0.156 | 164 |
| | Marine food web | Directed | 134 | 598 | 4.46 | 1.000 | 2.05 | – | 0.16 | 0.23 | −0.263 | 160 |
| | Freshwater food web | Directed | 92 | 997 | 10.84 | 1.000 | 1.90 | – | 0.20 | 0.087 | −0.326 | 209 |
| | Neural network | Directed | 307 | 2 359 | 7.68 | 0.967 | 3.97 | – | 0.18 | 0.28 | −0.226 | 323, 328 |

**Table 8.1: Basic statistics for a number of networks.** The properties measured are: type of network, directed or undirected; total number of vertices $n$; total number of edges $m$; mean degree $c$; fraction of vertices in the largest component $S$ (or the largest weakly connected component in the case of a directed network); mean geodesic distance between connected vertex pairs $\ell$; exponent $\alpha$ of the degree distribution if the distribution follows a power law (or "–" if not; in/out-degree exponents are given for directed graphs); clustering coefficient $C$ from Eq. (7.41); clustering coefficient $C_{WS}$ from the alternative definition of Eq. (7.44); and the degree correlation coefficient $r$ from Eq. (7.82). The last column gives the citation(s) for each network in the bibliography. Blank entries indicate unavailable data.

# Components

- In an unidirected network, there is typically a large component that fills most of the network
- Very often over $90\%$
- Sometimes, it is $100\%$, e.g. the Internet
- Sometimes it depends also on how we collect data

# Components in a directed network

- Weakly connected components correspond to components in an undirected network, i.e. we simply ignore link directions
- Otherwise, we have strongly connected components with corresponding in- and out-components
- Apart from the largest scc we have also a number of smaller ones with their in- and out-components
- Typically, all components form a so-called "bow-tie" model

# Components in a directed network

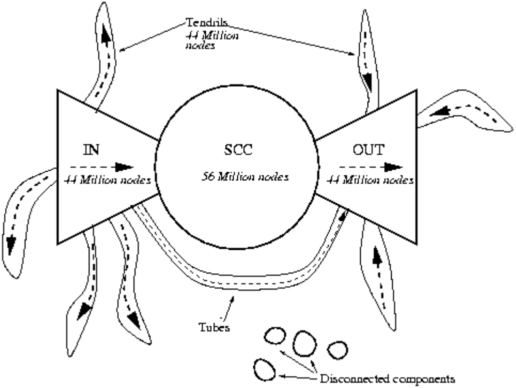



Figure: Bow-tie model of the Web graph

# Small-worlds

- In many networks the typical network distance between nodes is very small
- This phenomenon was first observed in the letter-passing experiment by Milgram
- It is called *small-world effect*
- Typically, the average network distance $\ell$ scales as $\log n$

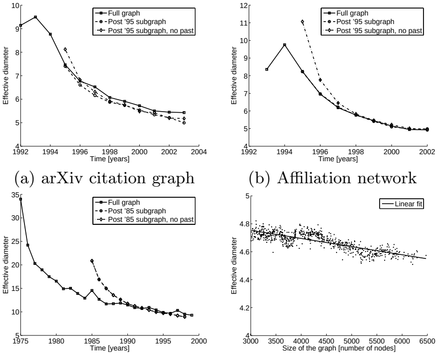
# Diameter

- Sometimes we are also interested in the network diameter
- The extreme of the distance distribution, i.e. the longest shortest path in the network
- In many networks, the core of the network is very dense with the average network distance scaling as $\log \log n$
- Whereas at the periphery the diameter scales as $\log n$

# Effective diameter

- Effective diameter, or 90-percentile effective diameter, i.e. 90% of shortest paths is smaller than the effective diameter
- Graphs over Time: Densification Laws, Shrinking Diameters and Possible Explanations by Leskovec et al.
- The empirical analysis has shown that when the networks grow the diameter becomes smaller

# Effective diameter




(a) arXiv citation graph
(b) Affiliation network
(c) Patents
(d) AS

Figure: Shrinking diameter

# Degree distributions

- Frequency distribution of node degrees
- One of the most fundamental properties of networks
- $p_k$ is the fraction of nodes in a network that has degree $k$
- $p_k$ is also a probability that a randomly chosen node has a degree $k$
- Typically, we visualize a distribution with a histogram

# Degree distributions

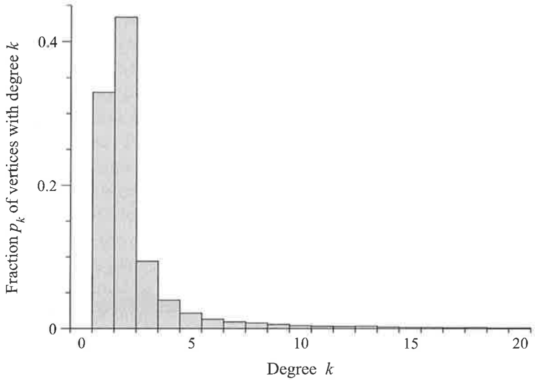



Figure: Degree distributions of the Internet graph at the level of autonomous systems

# Degree distributions

- Most of the nodes have small degrees: one, two, or three
- There is a *tail* to the distribution corresponding to the high-degree nodes
- The plot cuts off but the tail is much longer
- The highest degree node is connected to about 12% of other nodes
- Such well-connected nodes are called *hubs*

# Degree distributions

- It turns out that most of the real-world networks have such long-tailed distributions
- Such distributions are called *right-skewed*
- For directed networks we have two distributions
- In-degree and out-degree distribution

# Degree distributions

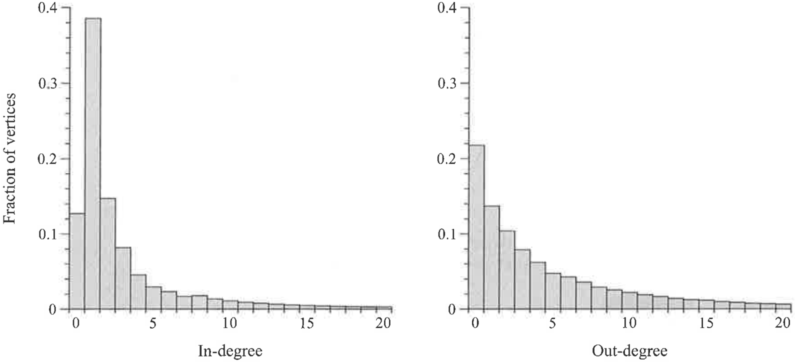



Figure: Degree distributions on the Web, from Broder et al.

# Power laws and scale-free networks

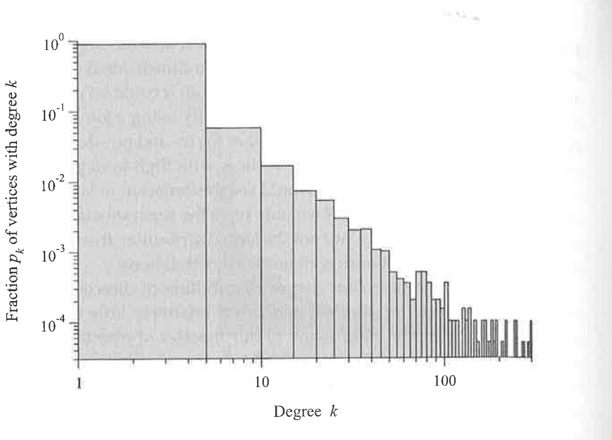



Figure: Degree distributions of the Internet graph on logarithmic scales

# Power laws

- The degree distribution on logarithmic scales follows roughly a straight line

$$\ln p_k = -\alpha \ln k + c \tag{1}$$

- $\alpha$ and $c$ are constants

$$p_k = Ck^{-\alpha} \tag{2}$$

- $C = e^c$ is another constant

# Power laws

- Distributions of this form that vary as a power of $k$ are called *power laws*
- This is a common pattern seen in many different networks
- The constant $\alpha$ is called the *exponent* of the power law
- Typical values are in the range: $2 \leq \alpha \leq 3$

# Power-law (Zipf) random variable

- Power-law distribution is a very commonly occurring distribution
- Word occurences in natural language
- Friendships in a social network
- Links on the web
- PageRank, etc.

# Power-law (Zipf) random variable

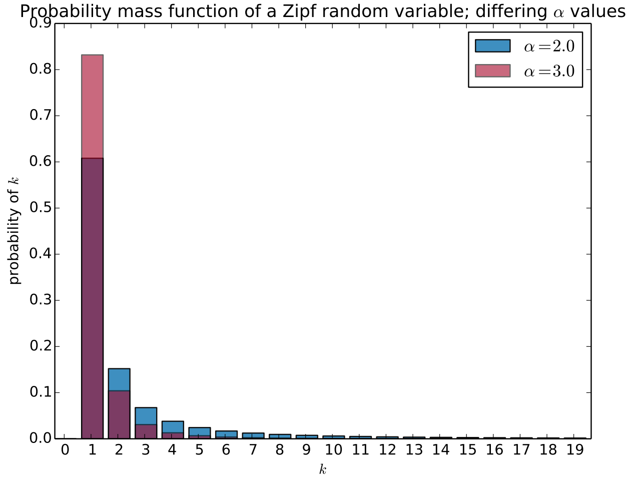
## PMF

$$p(k) = \frac{k^{-\alpha}}{\zeta(\alpha)}$$

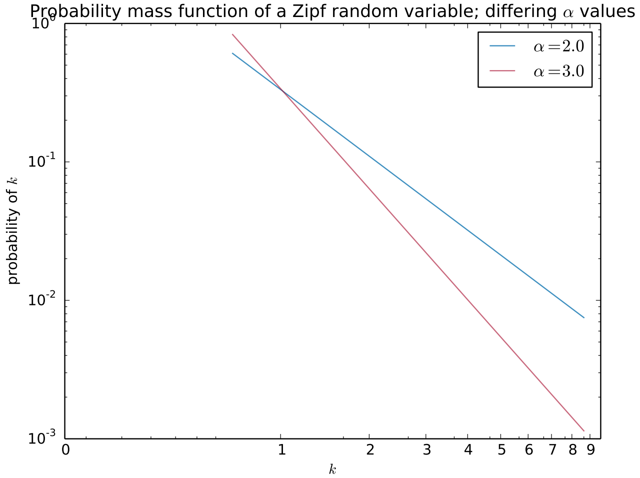
- $k \in \mathbb{N}$, $k \geq 1$, $\alpha > 1$
- $\zeta(\alpha)$ is the Riemann zeta function

$$\zeta(\alpha) = \sum_{k=1}^{\infty} k^{-\alpha}$$

# Power-law (Zipf) random variable



# Power-law (Zipf) random variable



# Power-law (Pareto) random variable

- Power-law distribution is a very commonly occurring distribution
- 80%-20% rule
- Wealth distribution
- The sizes of the human settlements
- File size of internet traffic, etc.

# Power-law (Pareto) random variable

## PDF

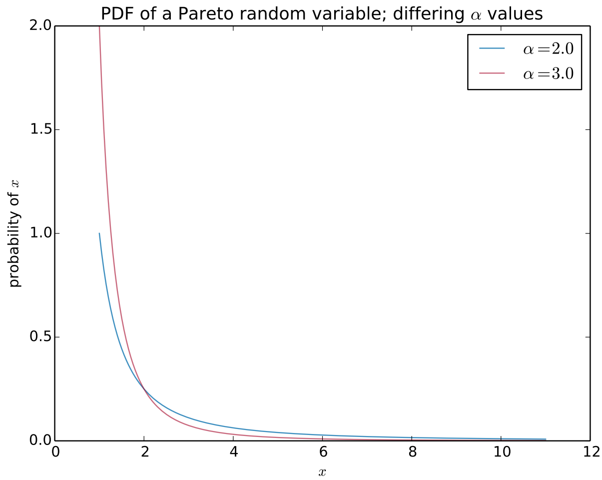
$$f(x) = \begin{cases} (\alpha - 1)\frac{x_{min}^{\alpha-1}}{x^{\alpha}}, x \geq x_{min} \\ 0, x < x_{min} \end{cases}$$

- $\alpha > 1$ is the exponent of the power-law distribution

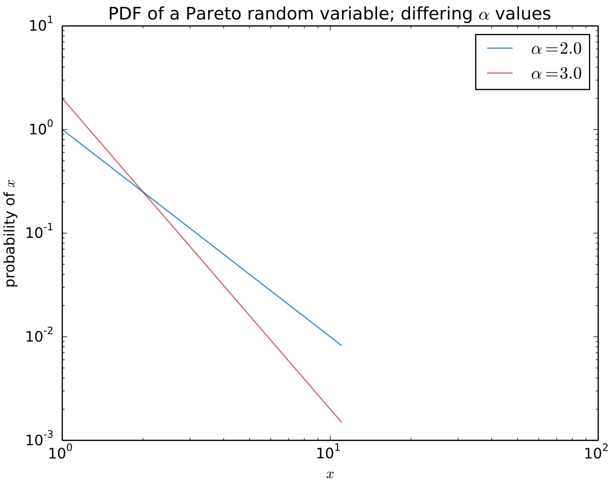
## CDF

$$f(x) = \begin{cases} 1 - (\frac{x_{min}}{x})^{\alpha-1}, x \geq x_{min} \\ 0, x < x_{min} \end{cases}$$

# Power-law (Pareto) random variable



# Power-law (Pareto) random variable



# Power laws

- Degree distributions do not follow power law equation over their entire range
- For example, for small $k$ we typically observe some deviation
- Thus, power laws are typically observed in the tail for high degrees
- Sometimes, there is also deviation in the tail because there is some cut-off that limits the maximum degree of nodes
- Network with power law degree distributions are called *scale-free* networks

# Detecting power laws

- Another common solution to visualizing power laws is to construct *cumulative distribution function*

$$P_k = \sum_{k'=k}^{\infty} p_{k'} \tag{3}$$

- $P_k$ is the fraction of nodes that have degree $k$ or higher

# Detecting power laws

- Suppose the degree distribution $p_k$ follows power law in the tail
- $p_k = Ck^{-\alpha}$, for $k \geq k_{min}$, for some $k_{min}$. Then for $k \geq k_{min}$:

$$P_k = \sum_{k'=k}^{\infty} k'^{-\alpha} \simeq C \int_{k}^{\infty} k'^{-\alpha} \mathrm{d}k' = \frac{C}{\alpha - 1} k^{-(\alpha-1)} \tag{4}$$

- Approximation of the sum by the integral is possible if we assume $\alpha > 1$ and is reasonable since the power law slowly varies for large $k$

# Detecting power laws

- Thus, cumulative degree distribution is also a power law but with an exponent $\alpha - 1$
- We can visualize the cumulative degree distribution on log-log scales and look for the straight line behavior
- This has some advantages over visualizing $p_k$
- E.g. we do not need to bin the histogram and throw away information

# Cumulative degree distributions

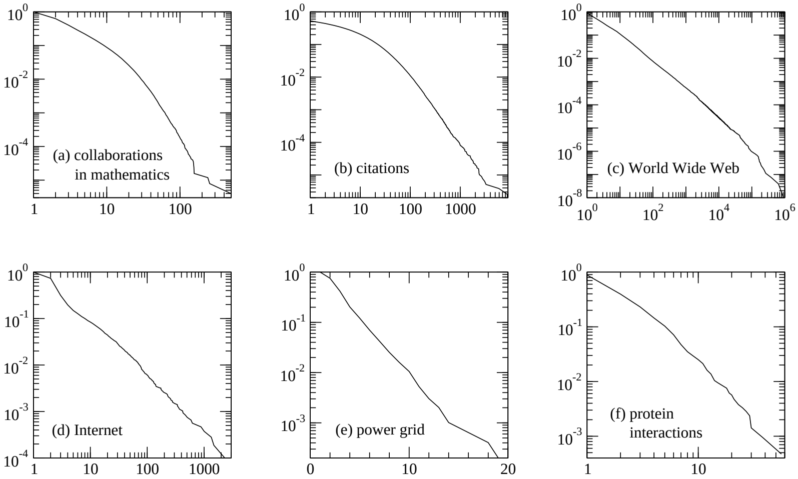



Figure: Cumulative degree distributions on logarithmic scales

# Cumulative degree distributions

- Cumulative degree distribution is easy to calculate
- The number of nodes greater or equal to that of the $r$th-highest degree is $r$
- The fraction of nodes with degree greater or equal to that of the $r$th-highest degree is $r/n$ and that is $P_k$
- Thus, we calculate degrees, sort them in descending order and then number them from 1 to $n$
- These numbers are *ranks* $r_i$ and we plot $\frac{r_i}{n}$ as a function of $k_i$

# Cumulative degree distributions

| Degree $k$ | Rank $r$ | $P_k = \frac{r}{n}$ |
|---|---|---|
| 4 | 1 | 0.1 |
| 3 | 2 | 0.2 |
| 3 | 3 | 0.3 |
| 2 | 4 | 0.4 |
| 2 | 5 | 0.5 |
| 2 | 6 | 0.6 |
| 2 | 7 | 0.7 |
| 1 | 8 | 0.8 |
| 1 | 9 | 0.9 |
| 1 | 10 | 1.0 |

Table: Example of cumulative degree distribution for degrees $\{0,1,1,2,2,2,2,3,3,4\}$

# Cumulative degree distributions

- Cumulative distribution have some disadvantages
- Successive points on a cumulative plot are not independent
- It is not valid to extract the exponent by fitting the slope of the line
- E.g. least squares method assumes independence of between the data points
- Also, which line to fit?

# Parameter estimation

- It is better to calculate $\alpha$ directly from the data

$$\alpha = 1 + N \left[ \sum_i \ln \frac{k_i}{k_{min} - \frac{1}{2}} \right]^{-1} \quad (5)$$

- where, $k_{min}$ is the minimum degree for which the power low holds and $N$ is the number of nodes with $k \geq k_{min}$

# Parameter estimation

- Statistical error

$$\sigma = \frac{\alpha - 1}{\sqrt{N}} \tag{6}$$

- The derivation is based on *maximum likelihood* techniques
- Power law distributions in empirical data by Clauset et al.
- http://tuvalu.santafe.edu/~aaronc/powerlaws/

# Likelihood

- We observe some data, e.g. number of heads in $m$ experiments with $n$ coin flips
- We **choose** a probabilistic model to describe the dataset
- E.g. a Binomial r.v. with parameters $(p, n)$
- $p$ is the probability of heads on a single coin flip

**PMF**

$$p(x) = \binom{n}{x}(1-p)^{n-x}p^x \tag{7}$$

# Likelihood

- Let us denote with $X_1, \ldots, X_m$ r.v. associated with our $m$ experiments
- Each of them is a Binomial r.v. with parameters $(p, n)$
- They are mutually independent
- Independent and identically distributed (i.i.d.)

# Likelihood

- We are interested in probability of observing the results of our $m$ experiments
- For a single experiment:

**Probability of a single experiment**

$$p(x_i) = \binom{n}{x_i}(1-p)^{n-x_i}p^{x_i} \tag{8}$$

# Likelihood

- For all $m$ experiments (since experiments are i.i.d. r.v.)

**Probability of all experiments**

$$p(x_1, \ldots, x_m | p) = \prod_{i=1}^{m} \binom{n}{x_i} (1-p)^{n-x_i} p^{x_i} \tag{9}$$

- This probability is called **likelihood**
- It is the probability of data given the parameter $p$
- Another name is likelihood function (function of parameter $p$)

# Log-likelihood

- Typically, we take a logarithm and work with logs since it simplifies the analysis

## Log-likelihood

$$\mathcal{L}(p) = ln(\prod_{i=1}^{m} \binom{n}{x_i} (1-p)^{n-x_i} p^{x_i}) \quad (10)$$

$$= \sum_{i=1}^{m} (ln\binom{n}{x_i} + (n-x_i)ln(1-p) + x_i ln(p)) \quad (11)$$

$$= \sum_{i=1}^{m} ln\binom{n}{x_i} + ln(p)\sum_{i=1}^{m} x_i + ln(1-p)(mn - \sum_{i=1}^{m} x_i) \quad (12)$$

# Maximum Likelihood Estimation (MLE)

- Now, we are interested in $p$ that most likely generated the data
- The data are most likely to have been generated by the model with $p$ that maximizes the log-likelihood function
- Setting $\frac{d\mathcal{L}}{dp} = 0$ and solving for $p$ we obtain the *maximum likelihood estimate*

## MLE

$$\frac{d\mathcal{L}}{dp} = \frac{1}{p}\sum_{i=1}^{m} x_i - \frac{1}{1-p}(mn - \sum_{i=1}^{m} x_i) = 0 \quad (13)$$

$$p = \frac{\sum_{i=1}^{m} x_i}{mn} = \frac{1}{m}\sum_{i=1}^{m} \frac{x_i}{n} \quad (14)$$

# Parameter estimation

- We consider the continuous power law distribution

$$p(x) = \frac{\alpha - 1}{x_{min}} \left( \frac{x}{x_{min}} \right)^{-\alpha} \tag{15}$$

- Given a data set with $n$ observations $x_i > x_{min}$ we would like to know the value of $\alpha$ that is most likely to have generated the data

# Parameter estimation

- The probability that the data are drawn from the model

$$p(x|\alpha) = \prod_{i=1}^{n} \frac{\alpha - 1}{x_{min}} \left( \frac{x_i}{x_{min}} \right)^{-\alpha} \tag{16}$$

- This probability is called *likelihood* of the data given model

# Parameter estimation

- The data are most likely to have been generated by the model with $\alpha$ that maximizes this function
- Commonly, we work with *log-likelihood* $\mathcal{L}$
- $\mathcal{L}$ has the maximum at the same place likelihood

$$\mathcal{L} = \ln p(x|\alpha) = \ln \prod_{i=1}^{n} \frac{\alpha - 1}{x_{min}} \left( \frac{x_i}{x_{min}} \right)^{-\alpha} \tag{17}$$

## Parameter estimation

$$\mathcal{L} = n\ln(\alpha - 1) - n\ln x_{min} - \alpha \sum_{i=1}^{n} \ln \frac{x}{x_{min}} \tag{18}$$

- Setting $\frac{\partial \mathcal{L}}{\partial \alpha} = 0$ and solving for $\alpha$ we obtain the *maximum likelihood estimate*

$$\hat{\alpha} = 1 + n \left[ \sum_{i=1}^{n} \ln \frac{x_i}{x_{min}} \right]^{-1} \tag{19}$$

# Properties of power law distributions

- Normalization
- The constant $C$ that appears in the power law equation is determined by the normalization requirement

$$\sum_{k=1}^{\infty} p_k = 1 \tag{20}$$

- $k^{-\alpha} = \infty$, for $k = 0$ and therefore we start at $k = 1$

# Properties of power law distributions

$$C \sum_{k=1}^{\infty} k^{-\alpha} = 1 \tag{21}$$

$$C = \frac{1}{\sum_{k=1}^{\infty} k^{-\alpha}} = \frac{1}{\zeta(\alpha)} \tag{22}$$

- $\zeta(\alpha)$ is the Riemann zeta function

# Properties of power law distributions

- Correctly normalized power law distribution for $k > 0$ and $p_0 = 0$

$$p_k = \frac{k^{-\alpha}}{\zeta(\alpha)} \tag{23}$$

- If the power law behavior holds only for $k > k_{min}$ we obtain (with $\zeta(\alpha, k_{min})$ being incomplete zeta function)

$$p_k = \frac{k^{-\alpha}}{\sum_{k=k_{min}}^{\infty} k^{-\alpha}} = \frac{k^{-\alpha}}{\zeta(\alpha, k_{min})} \tag{24}$$

# Properties of power law distributions

- Alternatively, we can approximate the sum with an integral

$$C \simeq \frac{1}{\int_{k_{min}}^{\infty} k^{-\alpha} \mathrm{d}k} = (\alpha - 1) k_{min}^{\alpha - 1} \tag{25}$$

$$p_k \simeq \frac{\alpha - 1}{k_{min}} \left( \frac{k}{k_{min}} \right)^{-\alpha} \tag{26}$$

# Properties of power law distributions

- Top-heavy distributions
- Another interesting property is the fraction of links that connect to the nodes with the highest degrees
- For a pure power law $W$ is a fraction of links attached to a fraction $P$ of the highest degree nodes

$$W = P^{\frac{\alpha-2}{\alpha-1}} \tag{27}$$

# Properties of power law distributions

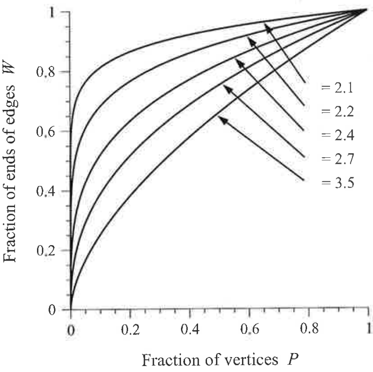



Figure: Lorenz curves for power law networks

# Properties of power law distributions

- The curves have a very fast initial increase (especially if $\alpha$ is slightly over 2)
- This means that a large fraction of links is connected to a small fraction of the highest degree nodes
- For example, in-degrees on the Web have $k_{min} = 20$ and $\alpha = 2.2$
- For $P = 0.5$ we have $W = 0.89$, for $W = 0.5$ we have $P = 0.015$

# Properties of power law distributions

- These calculations assume perfect power law
- We can still calculate $W$ and $P$ directly from the data
- For example, on the Web for $W = 0.5$ we have $P = 0.011$
- Similarly, in citation networks for $W = 0.5$ we have $P = 0.083$

# Centralities

- Eigenvector centralities have often a highly right-skewed distributions
- Also, variants of the eigenvector centralities such as PageRank exhibit often power law behavior
- E.g. the Internet, WWW, or citation networks
- Betweenness centrality also tends to have right-skewed distributions

# Centralities

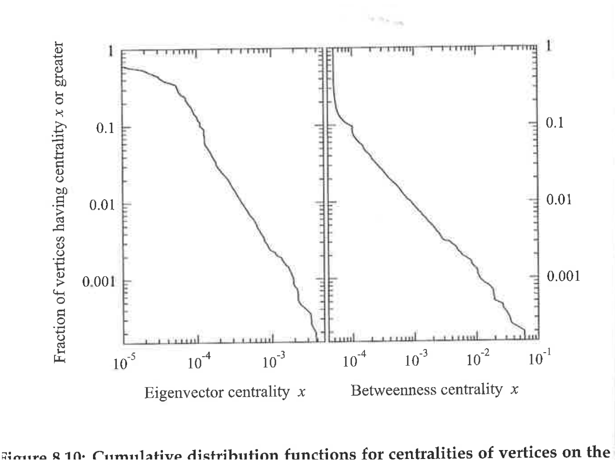



Figure: Cummulative distibutions of centralities on the Internet

# Centralities

- An exception to this pattern is closeness centrality
- Values for closeness centralities are limited by 1 at the lower end and $\log n$ at the upper end
- Therefore their distributions cannot have a long tail
- Typically, closeness centrality distributions are multimodal, whit multiple peaks and dips

# Centralities

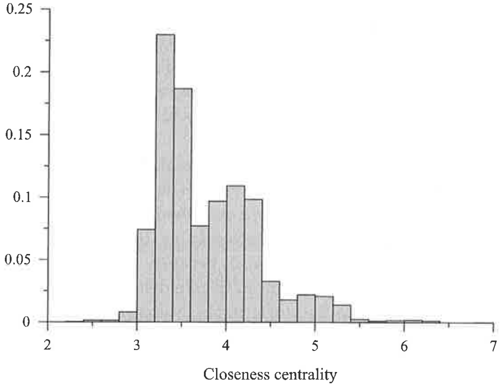



Figure: Histogram of closeness centralities on the Internet

# Global clustering coefficient

- The clustering coefficient measures the average probability that two neighbors of a node are themselves neighbors
- It measures the density of triangles in the networks
- In real networks the clustering coefficient takes values in the order of tens of percent, e.g. 10% or even up to 60%
- This is much larger than what we would expect if the links are created by chance, e.g. 0.01%
- E.g. in collaboration networks of physicists expectation is 0.23% but the real value is 45%

# Global clustering coefficient

- This large difference is indicative of social effects
- For example, it might be that people introduce the pairs of their collaborators to each other
- In social networks this process is called *triadic closure*
- An open triad of nodes is closed by the introduction of the last third link
- We can study the triadic closure processes directly if we have different version of datasets in time
- E.g. a study showed that it is much more likely (45 times) for people to collaborate in future if they had common collaborators in the past

# Global clustering coefficient

- In some networks we have the opposite phenomenon
- The expected value of clustering exceeds the observed one
- For example, on the Internet we measure 1.2% and the expected value is 84%
- Thus, on the Internet we have mechanisms that prevent forming of triangles
- On the Web the measured clustering coefficient is of the order of the expected one

# Global clustering coefficient

- It is not completely clear why different types of networks exhibit such different behaviors in respect to the clustering coefficient
- One theory connects these observations with the formation of communities in networks
- Social networks tend also to have positive degree correlations as opposed to other types of networks
- Thus, in social networks homophily and assortative mixing by degree plays a more important role than in other networks
- This tends to formation of communities and therefore the clustering coefficient becomes greater

# Local clustering coefficient

- Local clustering coefficient of a node $i$ is the fraction of neighbors of $i$ that are themselves neighbors
- In many networks there is a phenomenon that high degree nodes tend to have lower local clustering
- One possible explanation for this behavior is that nodes tend to form highly connected communities
- Communities of low degree nodes are smaller that work as small disconnected networks, i.e. cliques
- Probability that higher degree nodes form such huge cliques is rather small

# Local clustering coefficient

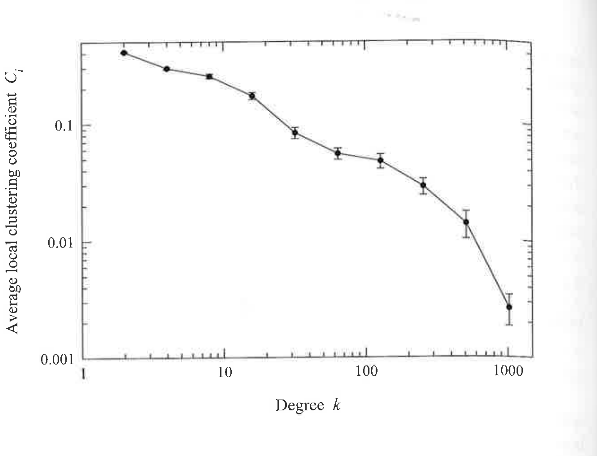



Figure: Local clustering as a function of degree on the Internet

# Assortative mixing by degree

- Assortative mixing by degree can be quantified by the correlation coefficient $r$
- Typically, $r$ is not of a large magnitude in real world networks
- There is clear tendency of social networks to have positive $r$ (homophily)
- Technological, information, biological networks tend to have negative $r$
- Simple graphs bias: the number of links between high-degree nodes is limited because they connect to low degree nodes
- Social networks: communities

# Network analysis project

- Software
- C++: SNAP http://snap.stanford.edu/
- Python: NetworkX http://networkx.github.io/
- Python wrapper for Boost: Graph-Tool http://graph-tool.skewed.de/
- Python, R, C: IGraph https://igraph.org/

# Network analysis project

- SNAP: http://snap.stanford.edu/
- KONECT: http://konect.uni-koblenz.de/
- Dataset of choice
- From SNAP or KONECT Web site
- Your own dataset