Freitag, 19. Oktober 2018 10:12

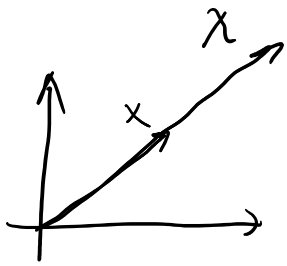
$x \in \mathbb{C}^n \quad x \neq 0$

$\lambda \in \mathbb{C}$

$A \in \mathbb{R}^{n \times n}$

$Ax = \lambda x$

$A(cx) = \lambda(cx)$



Freitag, 19. Oktober 2018 10:16

Characteristic Polynomial Equation

$$Ax = \lambda x$$

$$Ax - \lambda x = \vec{0}$$

$$Ax - \lambda I x = \vec{0}$$

$$\underbrace{(A - \lambda I)}_{\text{singular}} x = \vec{0}$$

$$\det(A - \lambda I) = 0$$

$$I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$n \times n$

$$\lambda_i = a + bi$$

$$\lambda_j = a - bi$$

Example 1

$$A = \begin{pmatrix} -1 & -1 \\ -5 & 3 \end{pmatrix} \qquad \lambda \underline{I} - A = \begin{pmatrix} \lambda+1 & 1 \\ 5 & \lambda-3 \end{pmatrix}$$

$$\det \begin{pmatrix} \lambda+1 & 1 \\ 5 & \lambda-3 \end{pmatrix} = (\lambda+1)(\lambda-3) - 5$$

$$= \lambda^2 + \lambda - 3\lambda - 3 - 5 = \lambda^2 - 2\lambda - 8$$

$$= \lambda^2 - 4\lambda + 2\lambda - 8 = \lambda(\lambda-4) + 2(\lambda-4)$$

$$= (\lambda+2)(\lambda-4)$$

$\lambda_1 = -2$

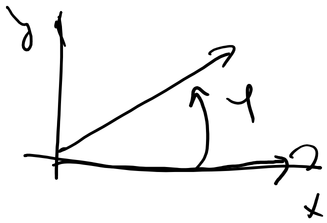
$\lambda_2 = 4$

$$A x_1 = \lambda_1 x_1 \qquad \begin{pmatrix} -1 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix} = \begin{pmatrix} -2x_1^{(1)} \\ -2x_1^{(2)} \end{pmatrix}$$

$$-x_1^{(1)} - x_1^{(2)} = -2x_1^{(1)} \Rightarrow x_1^{(1)} = x_1^{(2)}$$

$$-5x_1^{(1)} + 3x_1^{(2)} = -2x_1^{(2)} \qquad x_1 \doteq \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Example 2



$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}$$

$$\varphi = 90^\circ \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\det(\lambda I - A) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix} = \begin{pmatrix} i x_1^{(1)} \\ i x_1^{(2)} \end{pmatrix} \Rightarrow \begin{aligned} -x_1^{(2)} &= i x_1^{(1)} \\ x_1^{(1)} &= i x_1^{(2)} \end{aligned}$$

Freitag, 19. Oktober 2018 10:33

Properties of Eigenvalues and Eigenvectors of Real Symmetric Matrices

$A \in \mathbb{R}^{n \times n}$ $\quad A = A^T$

(1) $\lambda_i \in \mathbb{R} \quad \forall i$

(2) $x_i \perp x_j \quad \forall i \neq j$

Proof (1) $\quad c \in \mathbb{C} \quad \bar{c} \quad \overline{c_1 \cdot c_2} = \bar{c}_1 \cdot \bar{c}_2$

$$Ax = \lambda x \Leftrightarrow \overline{Ax} = \overline{\lambda x} \Leftrightarrow \bar{x}^T A = \bar{\lambda} \bar{x}^T \quad / \cdot x$$

$$\Leftrightarrow \bar{A}\bar{x} = \bar{\lambda} \cdot \bar{x} \Leftrightarrow \bar{x}^T A \cdot x = \bar{\lambda} \bar{x}^T \cdot x$$

$$\Leftrightarrow A\bar{x} = \bar{\lambda} \cdot \bar{x} \;/^T \Leftrightarrow \bar{x}^T \cdot \lambda x = \bar{\lambda} \bar{x}^T \cdot x$$

$$\lambda \cancel{\bar{x}^T x} = \bar{\lambda} \cancel{\bar{x}^T \cdot x}$$

$$\mathrm{im}(\lambda) = 0 \qquad \lambda = \bar{\lambda}$$

① $x_i \perp x_j \quad \forall i \neq j$

λ Multiplicity

$\boxed{\text{Algebr. multp} \geq 2}$

Geomet. multp. $= 1$

$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \ldots \neq \lambda_n$

$\lambda_i \neq \lambda_j \quad \forall i \neq j$

$A x_i = \lambda_i x_i \qquad A x_j = \lambda_j x_j$

$(\lambda_i x_i)^T \cdot x_j = \lambda_i x_i^T x_j$ (a)

$(\lambda_i x_i)^T \cdot x_j \overset{A}{=}$ (b)

$(A x_i)^T \cdot x_j = x_i^T A^T x_j = x_i^T (A x_j) = x_i^T \lambda_j x_j$

$\boxed{\lambda_i x_i^T x_j = \lambda_j x_i^T x_j}$

$\lambda_i \cdot c = \lambda_j \cdot c$

$\lambda_i = 1$

$\lambda_j = 2$

$2 \cdot c = 1 \cdot c$

$c = 0 \iff x_i^T x_j = 0 \iff x_i \perp x_j$

$e_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$B = A + \boxed{\begin{bmatrix} \varepsilon & & & 0 \\ & 2\varepsilon & & \\ & & \ddots & \\ 0 & & & n \cdot \varepsilon \end{bmatrix}} \qquad \varepsilon \to 0$$

Schur's Theorem

Spectral Theorem for Symm. Matrices

$A \in \mathbb{R}^{n \times n}$ $A = A^T$ $A = U \Lambda U^T$ → orthogonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$U \cdot U^T = I$

↓

orthonormal matrix

$$U = \begin{bmatrix} x_1 & \dots & x_n \\ & & \end{bmatrix}$$

$x_1, \dots, x_n$

Properties of orthonormal matrices

$v \qquad \sqrt{v^T v} \qquad \|v\|_2 = \|w\|_2$

$w = Uv$

$w^T w = (Uv)^T (Uv) = v^T \underbrace{U^T U}_{I} v = v^T I v = v^T v$

U is rotation

$A = U \Lambda U^T$

$+\varphi \qquad -\varphi$

$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Properties of eigenvalues

$$\mathrm{Tr}(A) = \sum_i \lambda_i$$

$$\mathrm{Tr}(A) = 0 = \sum_i \lambda_i$$

$$\det(A) = \prod_i \lambda_i$$

Freitag, 19. Oktober 2018 11:06

#Paths of a given length

A^2 , A^3 , A^r

$$A^r = (U \Lambda U^T)^r = \underbrace{(U \cdot \Lambda \overbrace{U^T) \cdot (U}^{I} \Lambda \overbrace{U^T) \ldots (U}^{I} \Lambda U^T)}_{r \text{ terms}}$$

$$= U \cdot \underbrace{\Lambda \cdot \Lambda \cdot \Lambda \cdot \ldots \Lambda}_{r \text{ terms}} \cdot U^T$$

$$= U \Lambda^r \cdot U^T$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \Lambda^r = \begin{bmatrix} \lambda_1^r & & 0 \\ & \ddots & \\ 0 & & \lambda_n^r \end{bmatrix}$$

$$\mathrm{Tr}(A^r) = \mathrm{Tr}(U \Lambda^r U^T) = \mathrm{Tr}(\overbrace{U U^T}^{I} \Lambda^r) = \mathrm{Tr}(\Lambda^r)$$

$$= \sum_i \lambda_i^r$$

Neumann Series, Geometric Series for Matr.

$i, j \quad [A^3]_{ij} = 10^5$

$= 3$

$\rightarrow$ measure of similarity

$\sum_p (\alpha A)^p$

$\sigma_1 = \alpha A + \alpha^2 A^2 + \alpha^3 A^3 + \dots + \alpha^p A^p + \dots \qquad p = ?$

$A = [2] \qquad \alpha < 1 \qquad p \to \infty$

$\alpha^p \to 0$

Freitag, 19. Oktober 2018 11:19

$$\sum_{r=0}^{\infty} \boxed{(\alpha A)^r} = (I - \alpha A)^{-1}$$

$$T = \alpha A$$

$$S_p = \sum_{r=0}^{p} (\alpha A)^r = \sum_{r=0}^{p} T^r$$

$$= I + T + T^2 + T^3 + \dots + T^p$$

$$(I - T) S_p = (I - T)(I + T + \dots + T^p)$$

$$\sum_{r=0}^{\infty} T^r$$

$$= I + \cancel{T} + \dots \cancel{+T^p} \cancel{-T} - \cancel{T^2} \dots - T^{p+1} = I - T^{p+1}$$

$$p \to \infty \quad T^{p+1} \to \vec{0} \iff (I - T) \sum_{r=0}^{\infty} T^r = I$$

$$T^{p+1} \xrightarrow[p \to \infty]{} 0 \qquad \square$$

Freitag, 19. Oktober 2018 11:26

Matrix Powers

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \qquad D^p = \begin{bmatrix} d_1^p & & 0 \\ & \ddots & \\ 0 & & d_n^p \end{bmatrix}$$

$$d_i < 1 \quad \forall i \qquad \sum_{p=0}^{\infty} d_i^p = \frac{1}{1-d_i} \qquad d_i < 1$$

$$A = U \Lambda U^T$$

$$\lim_{p \to \infty} A^p = U \Lambda^p U^T = \vec{0} \iff \lim_{p\to\infty} \Lambda^p \to \vec{0}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad |\lambda_i| < 1 \quad \forall i$$

$\rho(A)$ spectral radius

$$= \max_i |\lambda_i|$$

$$\rho(A) < 1$$

$$\rho(T) < 1 \iff \rho(\alpha A) < 1$$

$$\alpha |\lambda_n| < 1 \iff \alpha < \frac{1}{|\lambda_n|}$$

$$|\lambda_1| < \dots < |\lambda_n|$$

$$\sigma_{ij} = 1 + \alpha [A]_{ij} + \alpha^2 [A^2]_{ij} + \dots + \alpha^r [A^r]_{ij} + \dots$$

$$\alpha = 0.01 = 10^{-2}$$

$$\alpha^2 = 10^{-4}$$

$$\alpha^3 = 10^{-6} \qquad 10^5 \qquad = 0.1$$

$$\vdots$$

Rayleigh coefficients

$$R(x) = \frac{x^T A x}{x^T x} \qquad A \in \mathbb{R}^{n \times n} \qquad A = A^T$$

$$R(x) : \mathbb{R}^n \to \mathbb{R}$$

$$R(x) = R(cx) \qquad c \in \mathbb{R} \qquad c \neq 0$$

$$R(cx) = \frac{c x^T A c x}{c x^T c x} = \frac{\cancel{c^2} x^T A x}{\cancel{c^2} x^T x} = \frac{x^T A x}{x^T x} = R(x)$$

$$c = \frac{1}{\|x\|_2} \qquad R\left(\frac{x}{\|x\|_2}\right) = R(x)$$

$$\frac{x}{\|x\|_2} = x_{unit} \qquad R(x) = x_{unit}^T A x_{unit}$$

$$R(x) = x^T A x$$