Reyleigh coeff.

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$R(x)\ \mathbb{R}^n \to \mathbb{R}$ $R(x)$ = scalar = $\frac{x^T M x}{x^T x}$

$R(x) = x^T M x$ such that $x^T x = 1$

quadratic form

v is an eigenvec

$R(v_i) = \lambda_i$

$R(x) \in [\lambda_1, \lambda_n]$ s.t. $x^T x = 1$

$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$

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Extremal points $R(x)$

$R(x) = x^T M x \quad$ s.t. $\quad x^T x = 1$

$f(x): R \to R$

$R(x): R^n \to R$

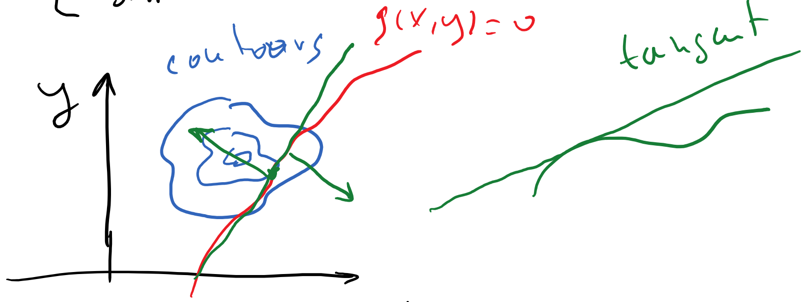
$$\nabla = \begin{bmatrix} \frac{\partial R}{\partial x_1} \\ \frac{\partial R}{\partial x_2} \\ \vdots \\ \frac{\partial R}{\partial x_n} \end{bmatrix}$$

$\nabla R = \emptyset$

Lagrange multipliers

$f(x,y)$

s.t. $g(x,y) < 0$



tangential to each other

$$r(t) = x(t)\vec{e_x} + y(t)\vec{e_y} \qquad r' = \frac{dx}{dt}\vec{e_x} + \frac{dy}{dt}\vec{e_y}$$

$$p(t) = f(x(t), y(t))$$

$$f(x, y) = c$$

$$p(t) = f(x(t), y(t)) = c$$

$$\frac{dp}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = 0$$

$$= \underbrace{\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix}}_{\nabla} \overbrace{\begin{bmatrix}\frac{dx}{dt} \\ \frac{dy}{dt}\end{bmatrix}}^{r'} = 0 \qquad \nabla \cdot r' = 0$$

$$\Leftrightarrow \nabla \perp r'$$

Gradients are parallel

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$$\nabla f = -\lambda \nabla g$$

Compare

$$\nabla f + \lambda \nabla g = 0$$

$$\nabla f = 0$$

Lagr.

↳ Lagrange mult.

$$\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) + \lambda \cdot g(\vec{x})$$

$$\nabla L = \nabla f + \lambda \nabla g = 0$$

$R(x) = x^T M x$ s.t. $\underbrace{x^T x = 1}$ $\Leftrightarrow$ $x^T x - 1 = 0$

$g(x) = 0$ $g(x) = x^T x - 1$

$g(x) = 1 - x^T x$

$L(x, \lambda) = x^T M x - \lambda (x^T x - 1)$

$x^T M x \Leftrightarrow c x^2$

$\nabla L = 0$

$2 c x$

$\nabla L = \cancel{2} M x - \cancel{2} \lambda x = \emptyset$

$\underbrace{\underset{n \times n \; \times \; n \times 1}{2 M x}}_{n \times 1}$

$\Leftrightarrow$

$M x = \lambda x$

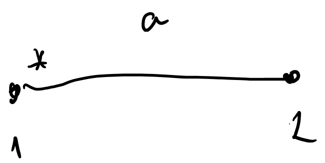
$x^T x$

n dectors

$[\lambda_1, \lambda_n]$

Link incidence Matrix $B \in R^{m \times n}$

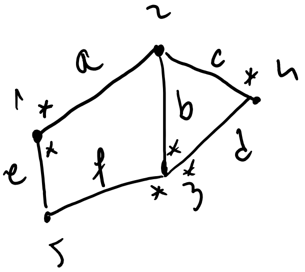
$$B_{ij} = \begin{cases} 1 & \text{if node } j \text{ is incident to link 1 at End 1} \\ -1 & \text{—————— " —————— at End 2} \\ 0 & \text{otherw} \end{cases}$$



$B_{a1} = 1$

$B_{a2} = -1$

$B_{ai} = 0$ for $i \neq 1 \neq 2$



$$B = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline a & 1 & -1 & 0 & 0 & 0 \\ b & 0 & -1 & 1 & 0 & 0 \\ c & 0 & -1 & 0 & 1 & 0 \\ d & & & & & \\ e & & & & & \\ f & & & & & \end{array} \leftarrow k$$

$B_{ki} B_{kj}$ $\quad i \neq j \quad$ $B_{ki} B_{kj} = -1$ iff $(i,j) \in E$

$B_{ki} B_{kj} = 0$ iff $(i,j) \notin E$

$i = j \quad$ $B_{ki} B_{kj} = 1$ iff link k conn. to i

$B_{ki} B_{kj} = 0$ otherwise

$\sum_k B_{ki} B_{kj} = \begin{matrix} -1 \\ 0 \end{matrix}$ $\begin{matrix} \text{iff conn.} \\ \text{otherw.} \end{matrix}$

$\sum_k B_{ki}^2 = k_i$ $\qquad \underset{(n \times m) \times (m \times n)}{B^T B}^{(n \times n)} = L$

$L = B^T B$

$Tr(L) = \sum_i \lambda_i$

$\lambda_1, \ldots, \lambda_n \geq 0$

v_i is eigenvector

$v_i^T L v_i = \lambda_i$ s.t. $v_i^T v_i = 1$

$v_i^T B^T B v_i = \underbrace{(Bv_i)^T}_{y^T} \underbrace{(Bv_i)}_{y} = y^T y \geq 0$

$= \lambda_i \geq 0$

Positive Semidefinite

$Tr(L) = \sum_i \lambda_i = 2m$

$\lambda_1 = \emptyset$

$L = D - A$

$L \cdot \vec{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ n times $= \vec{0} = 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ n times

$L \cdot \vec{1} = 0 \cdot \vec{1}$

$\Leftrightarrow \boxed{\lambda_1 = 0}$ $\quad v_1 = \vec{1}$ $\quad \leftarrow \vec{1} \cdot \frac{1}{\sqrt{n}}$

$\det(L) = 0 = \prod_i \lambda_i$

c Components

$$A = \begin{bmatrix} \boxed{1\ 1} & & \emptyset \\ & \boxed{1\ 1} & \\ \emptyset & & \boxed{1\ 1} \end{bmatrix} \qquad L = D - A \qquad L = \begin{bmatrix} \Box & & \emptyset \\ & \Box & \\ \emptyset & & \Box \end{bmatrix}$$

c times

$$\begin{bmatrix} \Box & & \emptyset \\ & \Box & \\ \emptyset & & \Box \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \emptyset \\ \emptyset \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \emptyset \\ \emptyset \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$v_1 \quad v_2 \quad v_3$

$\lambda_1 = 0 \quad \boxed{\lambda_2 = 0} \quad \lambda_3 = 0 \Rightarrow$ algebraic connectivity.

$L v_2 = \vec{0}$

$L v_2 = 0 \cdot v_2$

L as a linear operator

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B incidence matrix as a linear op. on vectors

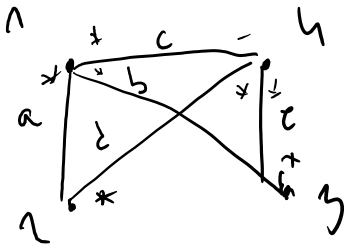
$B^{m \times n}$ nodes

$$
\begin{array}{c c}
 & \\
k \rightarrow & \begin{bmatrix} & & \\ 1 & & -1 \\ & & \end{bmatrix}
\end{array}
\overset{x}{\begin{bmatrix} x_i \\ x_j \end{bmatrix}_n} = \begin{bmatrix} x_i - x_j \\ \\ \\ \end{bmatrix}_m \; k^{th} \text{ ele}
$$

lines

k corr (i,j)

↳ Diff. operator.



$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

$$x = \begin{bmatrix} 10 \\ 6 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{Bx} = \begin{bmatrix} 5 \\ 9 \\ 10 \\ \\ \end{bmatrix}$$

$$B^T(Bx) = y \in \mathbb{R}^n$$

$$B^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ & & \cdots & & \end{bmatrix}$$

$$B^T B x = \begin{bmatrix} 24 \\ \vdots \\ \vdots \end{bmatrix}_n$$

second order diff.

$B^T B = L$ calcule second order diff.

$\lim_{\Delta t \to 0} \text{diff} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$

Sec. in the lim $\frac{d^2 x}{dt^2}$

Discrete version of second order par. deriv.

$$\nabla^2 = \boxed{\bar{\nabla} \cdot \bar{\nabla}} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2} + \ldots \frac{\partial^2 f}{\partial x_n^2} = \sum_i \frac{\partial f}{\partial x_i^2}$$

Laplacian : Graph Embeddings