Eigenvector Centrality

$$x_i^t = \sum_j A_{ij} x_j^{t-1}$$

$$x^0 = \vec{1}$$

$$x^1 = A \cdot x^0 = \vec{k} \quad \text{(Degree centrality)}$$

$$x^2 = A \cdot x^1$$

$$x^3 = A \cdot x^2$$

$$\vdots$$

$$x^t = A\, x^{t-1}$$

$$t \to \infty$$

Mittwoch, 31. Oktober 2018 12:37

A symmetric (undirected network)

$\lambda_i \in \mathbb{R}$ $\boxed{v_i \perp v_j \quad i \neq j}$

$$X^0 = \sum_i c_i v_i$$

$$X^t = A \cdot X^{t-1} = A \cdot A \cdot X^{t-2} = \underbrace{A \cdot A \cdot \ldots \cdot A}_{t} \cdot X^0 = A^t \cdot X^0$$

$$= A^t \sum_i c_i v_i = \sum_i c_i A^t v_i$$

$$= \sum_i c_i \underbrace{A \cdot A \cdot A \cdot \ldots \cdot \boxed{A v_i}}_{t} = \boxed{\sum_i c_i \lambda_i^t v_i}$$

$\lambda_i \cdot v_i$

$x^t = \sum_i c_i \lambda_i^t v_i$

$Tr(A) = 0 = \sum_i \lambda_i$

$\lambda_1 < \lambda_2 < \ldots < \lambda_n$

$x^t = \sum_i c_i \lambda_i^t v_i = \sum_i c_i \boxed{\lambda_n^t} \cdot \frac{\lambda_i^t}{\lambda_n^t} v_i$

$= \lambda_n^t \sum_i c_i \left(\frac{\lambda_i}{\lambda_n}\right)^t v_i \propto \boxed{v_n}$

$t \to \infty$ $\qquad \frac{\lambda_i}{\lambda_n} \to 0 \qquad i < n$

$\lim_{t \to \infty} x^t =$

$\rho(A) = |\lambda_1| \qquad |\lambda_1| \geq |\lambda_n|$

- v_n Perron vector $v_{n_i} \geq 0 \ \forall i$ (1)
- λ_n should have multiplicity 1 (2)
- $|\lambda_1| < \lambda_n \iff \rho(A) = \lambda_n$ (3)

v_n has all elements positive

$\lambda_1 \quad w \qquad w^T w = 1$

$$\boxed{|\lambda_1| w^T w} = |\lambda_1 w^T w| = |w^T A w| = |\sum_{ij} w_i A_{ij} w_j|$$

$$\leq \sum_{ij} |w_i A_{ij} w_j| = \sum_{ij} A_{ij} |w_i \, w_j|$$

Triangs ineq.

$$x_i = |w_i| \qquad = x^T A x \leq \lambda_n$$

$$x^T x = 1 \iff \sum_i x_i^2 = \sum |w_i|^2 = \sum w_i^2 = 1$$

$$|\lambda_1| \leq \lambda_n$$

$\lambda_n \quad v \quad v^T v = 1$

$$\boxed{\lambda_n v^T v} = |\lambda_n v^T v| = |v^T A v| = |\sum_{ij} v_i A_{ij} v_j|$$

$$\leq \sum_{ij} |v_i A_{ij} v_j| = \sum_{ij} A_{ij} |v_i v_j| = y^T A y \leq \lambda_n$$

Triang. ineq.

$y_i = |v_i| \qquad y^T y = 1$

$$\lambda_n \leq y^T A y \leq \lambda_n$$

$$\Rightarrow \quad y^T A y = \lambda_n \quad \Rightarrow \quad y_i \geq 0$$

Perron's Theorem

$A \in \mathbb{R}^{n \times n}$ $\quad A > 0 \iff A_{ij} > 0 \quad \forall\, ij$

$\implies$ multiplicity of $\lambda_n = 1$

Neuman's series

$$\sigma = \sum_{r=0}^{\infty} (\alpha A)^r \qquad \alpha < \frac{1}{\lambda_n}$$

$$\sigma = I + \alpha A + \alpha^2 A^2 + \alpha^3 A^3 + \ldots \qquad \sigma > 0$$

$$\sigma = (I - \alpha A)^{-1} \qquad \sigma_{ij} > 0 \quad \forall\, ij$$

$\iff$ network is connected

$$\sigma = (\underline{I} - \alpha A)^{-1} \qquad \frac{1}{1 - \alpha \lambda_n}$$

$\sigma > 0 \iff$ network is connected

$$\boxed{A \qquad \lambda} \qquad \alpha A \qquad \alpha\lambda$$

$$\underline{I} - \alpha A \qquad 1 - \alpha\lambda$$

$$(\underline{I} - \alpha A)x = \underbrace{\underline{I}x}_{} - \alpha \underbrace{Ax}_{} = 1 \cdot x - \alpha \lambda x$$

$$= (1 - \alpha\lambda)x$$

$$A^{-1} \qquad 1/\alpha$$

$$Ax = \lambda x \quad / A^{-1} \iff \underbrace{A^{-1}A}_{1} x = \lambda A^{-1} x$$

$$\iff x = \lambda A^{-1} x$$

$$\iff \frac{1}{\lambda} x = A^{-1} x$$

$\sigma > 0 \iff$ network is connect

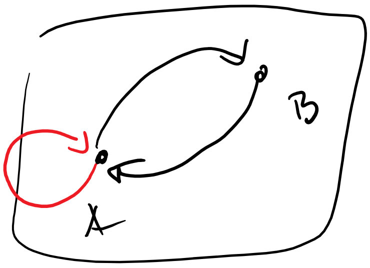
irreducible

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

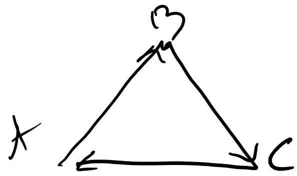
reducible

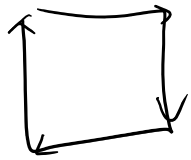
periodic network

Mittwoch, 31. Oktober 2018 13:27



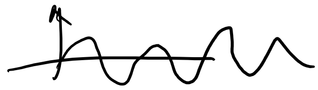
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$





$$(\lambda I - A) = \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix}$$

$$\det(\lambda I - A) = \lambda^2 - 1 \iff \lambda_{1,2} = \pm 1$$



ABABAXAB

Random Surfer ABABABAB _2 . . .

ABCABCABC _3 . . .

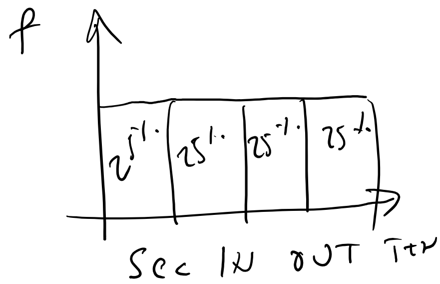
Perron-Frobenius Theorem

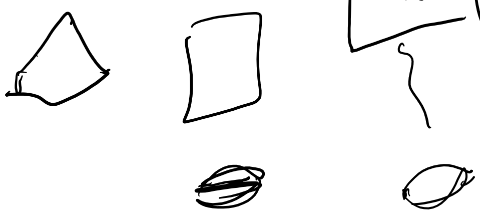
$Tr(A) > 0 \Rightarrow$ aperiodic $\Leftarrow\Rightarrow$ $\rho(A) = \lambda_n$

$\Rightarrow$ connected network, aperiodic network

directed network

Bow-tie



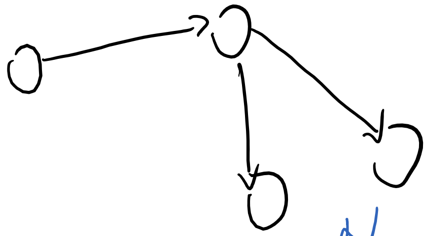


Random surfer

$1-\alpha$ prob. I'll follow a link at random

α prob I'll teleport

$1-\alpha = 0.85$

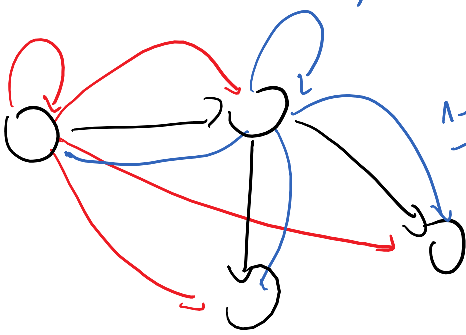


2003

30 billion eigen vector

>300

α/n



$\frac{1-\alpha}{k_i}$