Network Science (VU) (706.703)

Measuring Network Properties

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Outline

Introduction

- Once when we know the structure of the network we can calculate many useful quantities
- Such network analysis originates from *social network analysis*
- Mostly these ideas reflect some sociological concepts, such as influence, status, balance, ...
- However, today many of the methods from social network analysis are applied in computer science, physics, biology, and so on

Centrality

- One of the key topics in network science is *centrality*
- What are the most central nodes in a network?
- What are the most important nodes in a network?
- What are the most influential nodes in a network?

Centrality

- In different kind of networks different interpretation are possible
- E.g. in a social network the most central node might be the most popular person
- E.g. on the Web the most central node might be a page with the best quality of content in a specific field
- E.g. on the Internet the most central node might be a router with the highest bandwidth
- Thus, there are many possible definitions of importance and many possible interpretations and therefore there are many centrality measures

Centrality

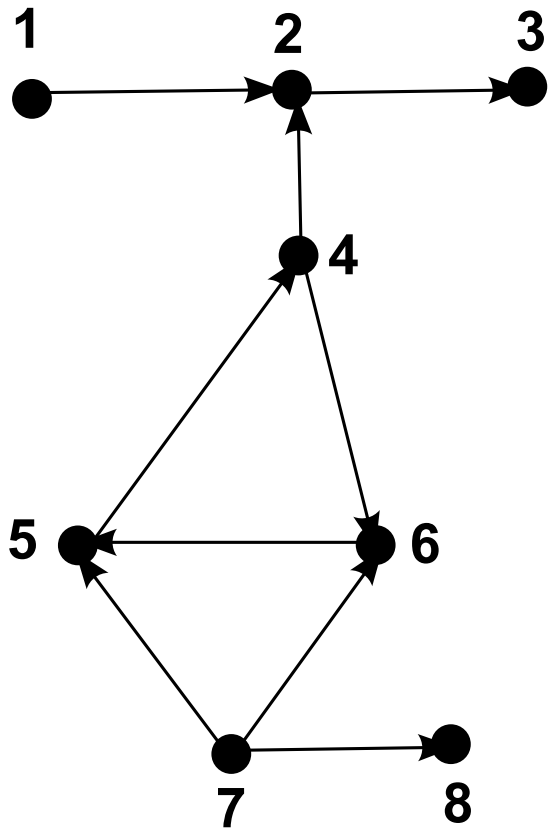
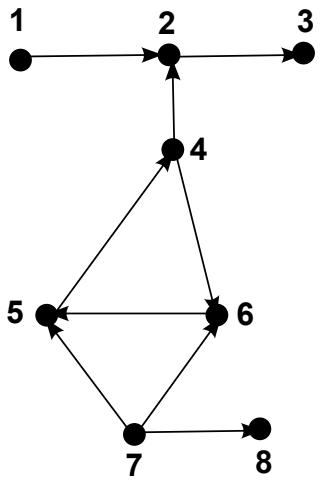


Figure: Sample network

Centrality



$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}$$

Degree centrality

- The simplest centrality measure is just the degree of a node
- In directed networks nodes have in- and out-degree and therefore there are two types of degree centrality
- In social networks persons that have high degree centrality might have better prestige, influence, access to information, ...
- In citation networks papers that have high in-degree centrality have a lot of citations
- This is a widely used metric for measuring the scientific impact of a paper

Degree centrality

- However, in many cases simple degree centrality is not enough
- E.g. a popular actor might have a high in-degree centrality, but is this a good proxy for measuring influence?
- Sometimes not only the number of links counts but also who are the neighbor nodes

Eigenvector centrality

- A natural extension of the degree centrality
- Degree centrality awards one centrality point for every neighbor a node has
- However, not all neighbors are equally important
- In many cases the importance of the node is increased by having connections to other nodes that are themselves important

Eigenvector centrality

- Basic concept of eigenvector centrality: not only count of neighbors is important but also the importance of the neighbors
- Degree centrality awards nodes with one centrality point for each neighbor
- Eigenvector centrality gives each node a score proportional to the sum of the scores of its neighbors
- Typically, we calculate eigenvector centralities iteratively

Eigenvector centrality

- We make an initial guess about the centrality x_i of each node i
- E.g. we set $x_i^0 = 1$ for all i
- Then we calculate a new iteration x_i^1 as the sum of the centralities of i's neighbors

$$x_i^1 = \sum_j A_{ij} x_j^0 \tag{2}$$

$$\mathbf{x}^1 = \mathbf{A}\mathbf{x}^0 \tag{3}$$

Eigenvector centrality

- In matrix form we have

$$\mathbf{x}^1 = \mathbf{A}\mathbf{x}^0 \tag{4}$$

- After t steps we have

$$\mathbf{x}^t = \mathbf{A}^t\mathbf{x}^0 \tag{5}$$

Eigenvector centrality

- We can write $\mathbf{x}^0$ as a linear combination of the eigenvectors $\mathbf{v}_i$ of the adjacency matrix (for appropriate choice of constants c_i)

$$\mathbf{x}^0 = \sum_i c_i \mathbf{v}_i \tag{6}$$

$$\mathbf{x}^t = \mathbf{A}^t \sum_i c_i \mathbf{v}_i = \sum_i c_i \mathbf{A}^t \mathbf{v}_i = \sum_i c_i \kappa_i^t \mathbf{v}_i = \kappa_1^t \sum_i c_i \left[\frac{\kappa_i}{\kappa_1}\right]^t \mathbf{v}_i \tag{7}$$

Eigenvector centrality

$$\mathbf{x}^t = \kappa_1^t \sum_i c_i \left[\frac{\kappa_i}{\kappa_1} \right]^t \mathbf{v}_i \tag{8}$$

- κ_i are eigenvalues, and κ_1 is the largest of themselves
- $\frac{\kappa_i}{\kappa_1} < 1$ for all $i > 1$
- When $t \to \infty$ $\frac{\kappa_i}{\kappa_1} \to 0$, for all $i > 1$
- When $t \to \infty$ $\mathbf{x}^t \to c_1 \kappa_1^t \mathbf{v}_1$

Eigenvector centrality

- In other words, the limiting vector of centralities is proportional to the leading eigenvector of the adjacency matrix
- In matrix form the centrality $\mathbf{x}$ satisfies:

$$\mathbf{A}\mathbf{x} = \kappa_1 \mathbf{x} \tag{9}$$

$$x_i = \frac{1}{\kappa_1} \sum_j A_{ij} x_j \tag{10}$$

Eigenvector centrality

- Thus, eigenvector centrality of a node can be large if a node has many neighbors or if it has important neighbors, or both
- If a person knows a lot of people (even if they are not important)
- Or if a person knows only a few people but in high places
- The eigenvector centralities are all non-negative
- If we chose $\mathbf{x}^0$ with all non-negative elements, multiplication by $\mathbf{A}$ can never introduce negative elements since all elements in $\mathbf{A}$ are non-negative

Eigenvector centrality

- Centralities are not normalized
- Typically, we are interested only in relative centralities of nodes
- We want to know which are important nodes and how their importance compares to others
- Absolute values are not needed

Eigenvector centrality

- For directed networks some complications arise
- Directed networks have an asymmetric adjacency matrix
- Asymmetric matrices have two sets of eigenvectors: the left and the right eigenvectors
- The right eigenvectors sum over in-coming links
- The left eigenvectors sum over out-going links

Eigenvector centrality

- We assume that the importance is given through links pointing to a node
- E.g. in citation networks citations of a paper
- E.g. on the Web links from other Web pages on a particular page
- The right eigenvectors sum over in-coming links
- Thus, the correct definition is same as for undirected case

$$x_i = \frac{1}{\kappa_1} \sum_j A_{ij} x_j \tag{11}$$

Centrality

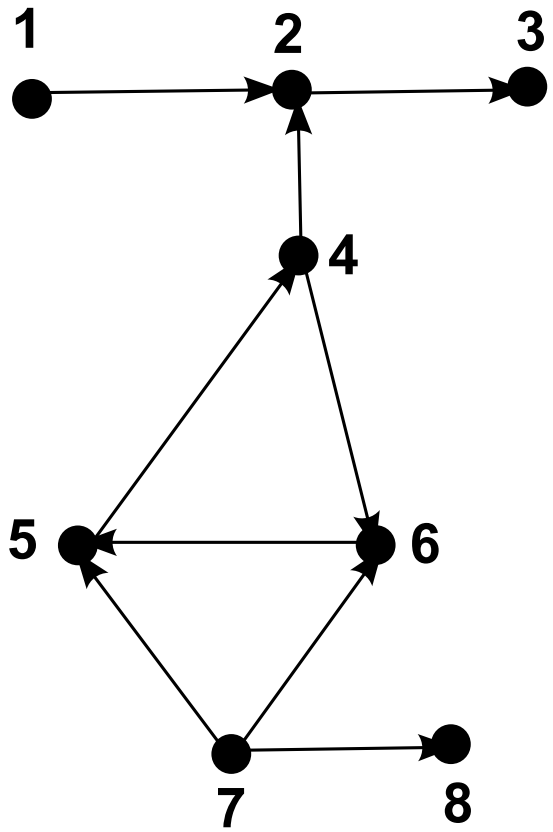


Figure: Sample network

Eigenvector centrality

- Nodes 7 and 1 have no incoming links
- Such nodes will always have eigenvector centrality zero
- This might be ok since they do not have incoming links
- However, node 8 has one incoming link but still centrality zero
- The link pointing to node 8 originates at node 7, and hence node 8 "inherits" centrality of node 7 which is zero
- This can propagate through many levels

Eigenvector centrality

- Only nodes that are in a strongly connected component or in its out-component can have eigenvector centralities larger than zero
- However, even nodes in an in-component might have many incoming links and be therefore important
- Acyclic networks have no strongly connected components and therefore all nodes have eigenvector centrality zero
- E.g. citation networks

Katz centrality

- One simple solution: we give each node a small amount of centrality

$$x_i = \alpha \sum_j A_{ij} x_j + \beta \tag{12}$$

- α and β are positive constants
- The first term is the normal eigenvector centrality and the second term is the "free" centrality

Katz centrality

- Even nodes with zero in-degree still get β centrality and can pass on this amount of centrality
- A node that has a high in-degree will always have a high centrality
- Also, nodes pointed by few other nodes with high centrality will also have a high centrality

Katz centrality

- In matrix form

$$\mathbf{x} = \alpha \mathbf{A}\mathbf{x} + \beta \mathbf{1} \tag{13}$$

$$\mathbf{x} = \beta (\mathbf{I} - \alpha \mathbf{A})^{-1} \mathbf{1} \tag{14}$$

Katz centrality

- Typically, we do not care about absolute values, thus β is unimportant
- We set $\beta = 1$

$$\mathbf{x} = (\mathbf{I} - \alpha \mathbf{A})^{-1} \mathbf{1} \tag{15}$$

Katz centrality

- The difference from standard eigenvector centrality is the free parameter α
- It weights the eigenvector term and the constant term
- Before we calculate the Katz centrality we have to chose a value for α
- If $\alpha \to 0$ then the eigenvector term disappears and only the constant term β remains
- By increasing α the centralities increase and there is a point at which they diverge

Katz centrality

- This happens when $(\mathbf{I} - \alpha\mathbf{A})^{-1}$, i.e. when $(\mathbf{I} - \alpha\mathbf{A})$ does not have an inverse
- I.e., $det(\mathbf{I} - \alpha\mathbf{A}) = 0$

$$det(\mathbf{A} - \alpha^{-1}\mathbf{I}) = 0 \tag{16}$$

Katz centrality

- But this is the characteristic equation with roots α^{-1} and these are eigenvalues of the adjacency matrix
- Thus, as α increases the determinant first becomes zero when $\alpha^{-1} = \kappa_1$
- After that the centralities diverge, i.e. whenever the determinant becomes zero again
- Thus, we should chose α less than $\frac{1}{\kappa_1}$ for the centralities to converge
- No further suggestions on choosing a value for α, i.e. chose it empirically

Note on the largest eigenvalue

- Symmetric matrix has all real eigenvalues
- Eigenvectors are orthogonal and form $\mathbb{R}^n$ vector basis
- According to Perron-Frobenius theorem an irreducible non-negative matrix has a real largest eigenvalue
- Other eigenvalues might be complex but come always in complex-conjugate form
- In both cases the leading eigenvector has all non-negative values

Centrality

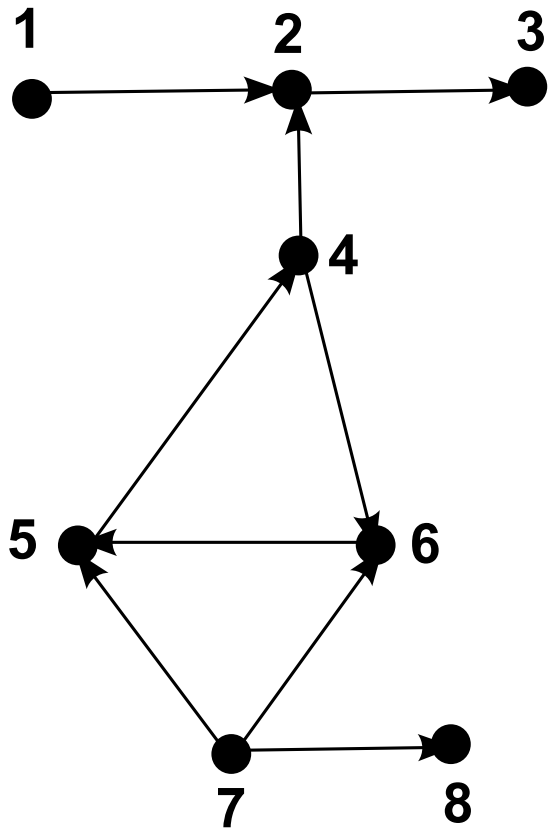


Figure: Sample network

Katz centrality

- Inverting a matrix has n^3 time complexity
- For large networks this is extremely slow
- Repeating the process many times $\mathbf{x}$ converges to a value close to the correct centrality

Katz centrality

- In each iteration step we have m multiplications as $\mathbf{A}$ has m non-zero elements
- Thus, the total time complexity is rm, where r is the number of iterations
- r depends on the network and α and no general guidelines exist
- Observe x_i, apply thresholds, etc.
- However, for large networks iteration is much faster then inverting the matrix

PageRank

- One problem with the Katz centrality
- If a node with high centrality points to many others then all of these nodes get also high centrality
- However, in many cases it means less if a node gets a link if it is only one of many
- E.g. Yahoo has many links but not all of the Web pages included in the directory are as important as Yahoo
- Better solution would be that a high centrality node passes only a fraction of its centrality to the neighbors

PageRank

- We can define a variation in which the centrality derived from neighbors is proportional to their centrality divided by their out-degree

$$x_i = \alpha \sum_j A_{ij} \frac{x_j}{k_j^{out}} + \beta \tag{17}$$

- If $k_i^{out} = 0$ we set $k_i^{out} = 1$, since $A_{ij} = 0$ for all i and the contribution of a node without outgoing links remains zero

PageRank

- In matrix form

$$\mathbf{x} = \alpha \mathbf{A} \mathbf{D}^{-1} \mathbf{x} + \beta \mathbf{1} \tag{18}$$

- $\mathbf{D}$ id the diagonal matrix with elements $D_{ii} = max(k_i^{out}, 1)$

PageRank

- Again, we do not care about absolute values, thus β is unimportant
- We set $\beta = 1$

$$\mathbf{x} = \alpha \mathbf{A} \mathbf{D}^{-1} \mathbf{x} + \mathbf{1} \tag{19}$$

PageRank

- Solving for $\mathbf{x}$

$$\mathbf{x} = (\mathbf{I} - \alpha \mathbf{A}\mathbf{D}^{-1})^{-1}\mathbf{1} = \mathbf{D}(\mathbf{D} - \alpha \mathbf{A})^{-1}\mathbf{1} \tag{20}$$

PageRank

- Again, PageRank has the free parameter α
- Before we calculate PageRank we have to chose a value for α
- By analogy with the Katz centrality, α should be less then inverse of the largest eigenvalue of $\mathbf{A}\mathbf{D}^{-1}$
- For undirected networks the largest eigenvalue is 1, thus α should be less than 1
- For directed networks the largest eigenvalue can be different than 1, but it is roughly of order 1
- Google uses $\alpha = 0.85$ (empirical choice)

Centrality

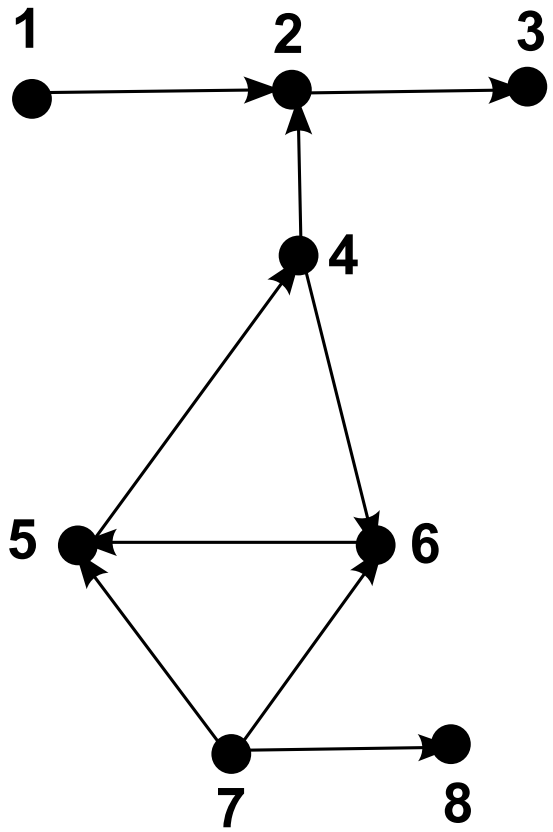


Figure: Sample network

Centrality

	with constant term	without constant term
divide by out-degree	$\mathbf{x} = \mathbf{D}(\mathbf{D} - \alpha\mathbf{A})^{-1}\mathbf{1}$ PageRank	$\mathbf{x} = \mathbf{A}\mathbf{D}^{-1}\mathbf{x}$ degree centrality
no division	$\mathbf{x} = (\mathbf{I} - \alpha\mathbf{A})^{-1}\mathbf{1}$ Katz centrality	$\mathbf{x} = \kappa_1^{-1}\mathbf{A}\mathbf{x}$ eigenvector centrality

Table: Comparison of centrality measures

Centralities Notebook

- Jupyter Notebook example
- http://kti.tugraz.at/staff/denis/courses/netsci/cent.ipynb

Closeness centrality

- A completely different measure is the *closeness centrality*
- It measures the average distance of a node to other nodes
- I.e. it measures the average shortest path length of a node to other nodes
- Let d_{ij} be the shortest path length between nodes i and j

$$\ell_i = \frac{1}{n} \sum_j d_{ij} \tag{21}$$

Closeness centrality

- That quantity is the average shortest path length of node i
- It is low for nodes that are separated by short distances from other nodes in the network
- E.g. such nodes might have better access to information, or more influence on the others in a social network
- This is however not a centrality measure since it gives low values to central nodes

$$C_i = \frac{1}{\ell_i} = \frac{n}{\sum_j d_{ij}} \tag{22}$$

Closeness centrality

- It is often used in network analysis, however it has some problems
- What is the dynamic range of the shortest path length in empirical networks
- Lower bound on d_{ij} is 1
- Upper bound is typically $logn$, which is e.g. 5, 6, or similar
- Thus, the range is small

Closeness centrality

- In practice the values of closeness centrality are very close to each other with differences in the trailing digits
- Very often you have huge number of nodes with the exact same closeness centrality
- The values are also very unstable
- Small changes in the network structure tend to have huge impact on the closeness centralities

Closeness centrality

- There is another problem
- ℓ_i is infinite for all i in a network with two or more components
- This can be solved by defining closeness centrality as the harmonic mean of distances between nodes

$$C_i = \frac{1}{n-1} \sum_{j(\neq i)} \frac{1}{d_{ij}} \tag{23}$$

- We have to exclude d_{ii} since this is zero

Betweenness centrality

- Betweenness centrality addresses dynamic processes that can take place on a network
- For example, suppose we a have a network with something flowing around
- E.g. messages, news, information, data packets
- A simple assumption is that objects will flow using shortest paths
- Then, the total number of messages that crosses a node is proportional to the number of shortest paths that each node lies on
- This is the betweenness centrality where more central nodes are more important for the communication processes that take place on the network

Betweenness centrality

- Let n_{st}^i be 1 if node i lies on a shortest path between s and t and zero if it does not, or there is no such path

$$x_i = \sum_{st} n_{st}^i \tag{24}$$

- This is the case where there is only one shortest path between s and t
- However, if we have more than one, e.g. g_{st}

$$x_i = \sum_{st} \frac{n_{st}^i}{g_{st}} \tag{25}$$

Betweenness centrality

- Betweenness centrality differs from the other centrality measures because it does not measure how well connected is a node
- Rather it measures its position in the network, and how much a node lies "between" other nodes
- Can you imagine a node that has high betweenness centrality but all other centralities are low
- I.e. it has low degree, it is on the periphery, etc.

Betweenness centrality

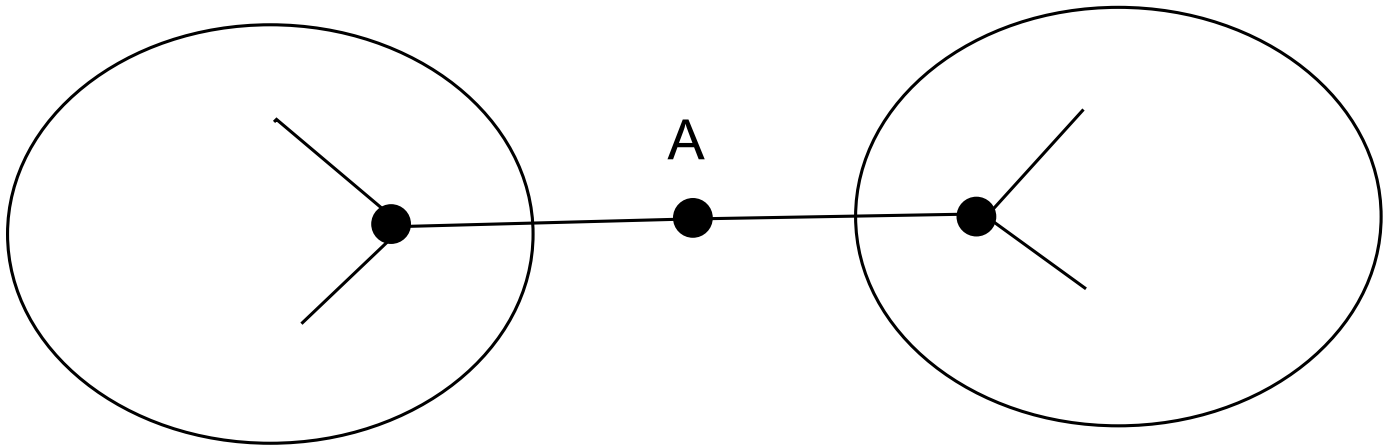


Figure: Node with high betweenness centrality

Betweenness centrality

- The range for betweenness centralities is rather large
- The maximum possible value for the betweenness centrality of a node is when the node lies on the shortest path between all pairs of nodes
- This occurs for the central node in a star network

Betweenness centrality

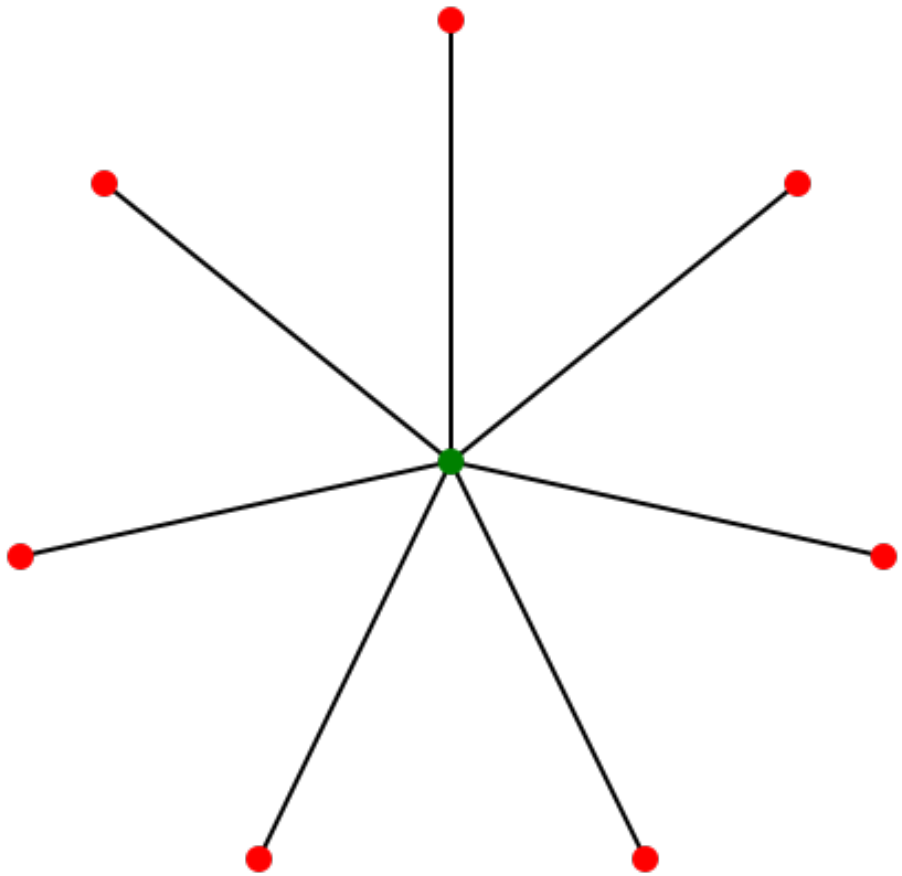


Figure: Star network

Betweenness centrality

- It lies on n^2 shortest paths between node pairs except for $n-1$ paths from the peripheral nodes to themselves
- Thus, the betweenness centrality of the central node is $n^2 - n + 1$
- The smallest possible value of the betweenness centrality in a connected network is when a node lies only on shortest paths to or from itself
- $n-1$ from such a node, $n-1$ to such a node and 1 to itself
- Thus, the minimum is $2n-1$

Betweenness centrality

- The ratio is $\frac{n^2-n+1}{2n-1}$
- Theoretically, it is approx. $\frac{1}{2}n$
- Moreover, it increases with n
- Also, differences between centralities of different nodes are larger and therefore the relative order of nodes is quite stable

Centrality: Project suggestions

- Correlations between various centralities
- Time evolution of centralities
- Rank-correlations
- Comparison of ranks with null models
- Configuration model: keep the degrees but create links at random

Clustering

- In social network a very important property is *transitivity*
- If "connected by a link" is transitive that would mean that if (u, v) and (v, w) then (u, w)
- The friend of a friend is also my friend
- Total transitivity occurs in a *clique*, i.e. in a fully connected network

Clustering

- More interesting is *partial transitivity*
- Social networks exhibit a high degree of partial transitivity
- (u, v) and (v, w) does not guarantee (u, w)
- But, it makes it much more likely

Clustering

- We can quantify it in the following way
- If (u, v) and (v, w) then we have a path of length two: uvw
- If also (u, w) then the path is closed forming a triangle
- In social networks literature this is called *closed triad*

Clustering

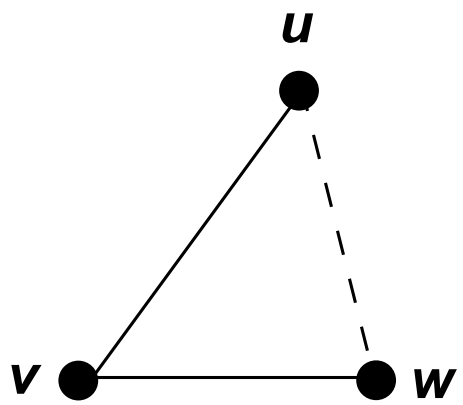


Figure: A closed triad

Clustering coefficient

- *Clustering coefficient* is the fraction of closed triads in the network, i.e. the fraction of paths of length two that are closed

$$C = \frac{(\text{number of closed paths of length two})}{(\text{number of paths of length two})} \tag{26}$$

$$C = \frac{(\text{number of triangles}) \times 6}{(\text{number of paths of length two})} \tag{27}$$

$$C = \frac{(\text{number of triangles}) \times 3}{(\text{number of connected triples})} \tag{28}$$

Clustering

- Social networks tend to have quite high values of the clustering coefficient
- E.g. actor collaborations $C = 0.2$, e-mail communication $C = 0.16$, etc.
- Technological and biological networks have smaller values of the clustering coefficient
- E.g. the Internet $C = 0.01$

Clustering

- What does it mean that these values for social networks are high?
- Clustering coefficient is the probability that two of my friends are also friends
- E.g. in a random network with the average degree c and n nodes, this probability is $\frac{c}{n}$
- E.g. for film actors this is 0.0003, for e-mail communication 0.00002
- Thus, the measured clustering coefficient is much larger than the estimation based on random network connections

Local clustering

- Clustering for a single node

$$C_i = \frac{(\text{number of connected pairs of neighbors of } i)}{(\text{number of pairs of neighbors of } i)} \tag{29}$$

Local clustering

- The number of pairs of neighbors of i equals $\frac{1}{2}k_i(k_i - 1)$
- It is the average probability that a pair of i's friends are friends with each other
- Local clustering depends on degree
- In most networks nodes with higher degrees have lower values of the local clustering coefficient

Local clustering

- The local clustering coefficient measures the existence of *structural holes*
- A lower value of the local clustering coefficient means that a lot of expected links between i's friends is actually missing
- Such structural holes improve the importance of the central node i
- E.g. the communication between friends can be controlled by i

Local clustering

- It is a local version of the betweenness centrality
- It measures the importance of the node for local communication whereas the betweenness centrality measures this at the global level
- Also, more central nodes have lower values for the local clustering coefficient
- In practice, betweenness and local clustering are strongly correlated (Burt, Structural Holes: The Social Structure of Competition)
- However, local clustering is faster to calculate

Global clustering

$$C_{WS} = \frac{1}{n} \sum_{i=1}^{n} C_i \tag{30}$$

- This is an alternative equation for calculating global clustering coefficient
- It gives typically different results as the previous equation
- It is dominated by the nodes of lower degrees

Reciprocity

- The clustering coefficient measures the frequency with which the loops of length three appear in a network
- In directed networks we can also concentrate of loops of length two
- A pair of nodes with links between them running in both directions
- The frequency of such loops is measured by *reciprocity*
- Reciprocity tells us how likely, on average, is that a node that you point to, points back to you
- Back link from a Web page, or a person you follow on twitter follows you

Reciprocity

- The reciprocity r is defined as the fraction of links that are reciprocated
- $A_{ij}A_{ji} = 1$, if and only if we have a link from i to j and from j to i
- $A_{ij}A_{ji} = 0$, otherwise

$$r = \frac{1}{m}\sum_{ij} A_{ij}A_{ji} = \frac{1}{m}Tr\mathbf{A}^2 \tag{31}$$

- $r \approx 0.57$ on the Web

Similarity

- How nodes can be similar to each other and how can we quantify this similarity
- Similarity can be calculated in many different ways even without the networks
- E.g. content-based similarity of text documents
- E.g. User similarities based on their profiles
- Here we are interested in measuring similarities based on the network properties such as links, degrees, etc.

Similarity

- There are two approaches to measuring similarity of nodes in a network
- *Structural similarity* is based on the number of the common neighbors
- *Regular similarity* is based on the similarity of their respective (not necessarily common) neighbors
- The distinction is similar to centrality measures
- Degree (neighbors count) vs. eigenvector centralities (recursion over the neighbors)

Structural similarity

- The most obvious measure of structural similarity is the number of common neighbors of two nodes

$$n_{ij} = \sum_k A_{ik} A_{kj} \tag{32}$$

- This is the ijth element of $\mathbf{A}^2$
- It is "cocitation" for undirected networks
- This is also the number of paths of length 2 between i and j

Structural similarity

- A simple count of common neighbors on its own is not a good measure of node similarity
- E.g. what does it mean that two nodes have 3 or 1000 common neighbors
- We need normalization
- We could normalize by dividing by the maximal number of common neighbors $(n-2)$
- However, this penalizes the nodes with low degrees, e.g. two nodes with degrees 3 and all three common neighbors will get a small amount of similarity in a large network

Cosine similarity

- The scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta \tag{33}$$

- where $|\mathbf{x}||\mathbf{y}|$ and θ is the angle between the two vectors

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|} \tag{34}$$

Cosine similarity

- We can regard ith and jth rows and columns of the adjacency matrix as vectors

$$\sigma_{ij} = \cos\theta = \frac{\sum_k A_{ik}A_{kj}}{\sqrt{\sum_k A_{ik}^2}\sqrt{\sum_k A_{jk}^2}} \tag{35}$$

Cosine similarity

- For simple networks: $A_{ij}^2 = A_{ij}$
- Then: $\sum_k A_{ik}^2 = \sum_k A_{ik} = k_i$

Definition

Cosine similarity

$$\sigma_{ij} = \frac{n_{ij}}{\sqrt{k_i k_j}} \tag{36}$$

- If any of degrees equals to zero then we set $\sigma_{ij} = 0$
- It always lies in the range 0 to 1

Pearson coefficients

- Another normalization possibility is to compare the number of common neighbors to the number of common neighbors if nodes select their neighbors randomly
- I.e. we compare the actual network structure with a random network structure
- We obtain in this way *Pearson correlation coefficient*

Pearson coefficients

- Let nodes i and j have degrees k_i and k_j respectively
- Now we first let i and then j to choose their neighbors randomly
- The probability that j selects one node that i already has chosen is equal to $\frac{k_i}{n}$
- In total: the expected number of common neighbors is $\frac{k_i k_j}{n}$
- We define now similarity as the difference between the actual number of common neighbors and the expected number if they chose their neighbors randomly

Pearson coefficients

$$\begin{aligned}
\sum_k A_{ik}A_{kj} - \frac{k_i k_j}{n} &= \sum_k A_{ik}A_{kj} - \frac{1}{n}\sum_k A_{ik}\sum_l A_{jl} \\
&= \sum_k A_{ik}A_{kj} - n\overline{A_i}\,\overline{A_j} \\
&= \dots \\
&= \sum_k (A_{ik} - \overline{A_i})(A_{kj} - \overline{A_j}) \qquad (37)
\end{aligned}$$

Pearson coefficients

- The last equation is n times covariance $\text{cov}(A_i, A_j)$ of the ith and jth row of the adjacency matrix
- It is positive if i and j have more common neighbors than what would be expected by chance
- It is negative if i and j have less common neighbors than what would be expected by chance
- It is zero if i and j have exactly as what would be expected by chance

Pearson coefficients

- We can normalize by the maximal value of the covariance which occurs when two set of quantities are the same
- Then covariance equals to variance of either sets σ_i^2, or σ_j^2, or $\sigma_i \sigma_j$
- Normalizing by this quantity we obtain the standard Pearson correlation coefficient
- $-1 \leq r_{ij} \leq 1$

Pearson coefficients

Definition

Pearson correlation coefficient

$$r_{ij} = \frac{\mathrm{cov}(A_i, A_j)}{\sigma_i \sigma_j} = \frac{\sum_k (A_{ik} - \overline{A_i})(A_{kj} - \overline{A_j})}{\sqrt{\sum_k (A_{ik} - \overline{A_i})^2} \sqrt{\sum_k (A_{kj} - \overline{A_j})^2}} \tag{38}$$

Regular similarity

- The structural similarity measures the extent to which two nodes share the same neighbors
- Regularly similar nodes are those that have neighbors that are similar
- These neighbors must not be shared
- The basic idea is to define a similarity score σ_{ij} such that i and j have high similarity if they have neighbors k and l that themselves have high similarity

Regular similarity

$$\sigma_{ij} = \alpha \sum_{kl} A_{ik} A_{jl} \sigma_{kl} \tag{39}$$

$$\boldsymbol{\sigma} = \alpha \mathbf{A} \boldsymbol{\sigma} \mathbf{A} \tag{40}$$

Regular similarity

- The formula does not give a high value for "self-similarity" σ_{ii}
- As a consequence this does not give a high similarity score to nodes that share neighbors
- If self-similarity is high this would also give a high similarity score to nodes with many common neighbors

$$\sigma_{ij} = \alpha \sum_{kl} A_{ik} A_{jl} \sigma_{kl} + \delta_{ij} \tag{41}$$

$$\boldsymbol{\sigma} = \alpha \mathbf{A} \boldsymbol{\sigma} \mathbf{A} + \mathbf{I} \tag{42}$$

Regular similarity

- What happens if we evaluate the formula iteratively with $\sigma^0 = 0$

$$\sigma^1 = \mathbf{I} \tag{43}$$
$$\sigma^2 = \alpha\mathbf{A}^2 + \mathbf{I} \tag{44}$$
$$\sigma^3 = \alpha^2\mathbf{A}^4 + \alpha\mathbf{A}^2 + \mathbf{I} \tag{45}$$

Regular similarity

- The pattern is clear
- In the limit of many iterations we get a sum over even powers of the adjacency matrix
- The elements of the rth power of $\mathbf{A}$ count the number of paths of length r between nodes
- Why should we count only paths of even length?

Regular similarity

- This leads to a better definition of regular similarity
- Nodes i and j are similar if i has a neighbor k that is itself similar to j
- Again we assume that nodes are similar to themselves

$$\sigma_{ij} = \alpha \sum_k A_{ik}\sigma_{kj} + \delta_{ij} \tag{46}$$

$$\boldsymbol{\sigma} = \alpha \mathbf{A}\boldsymbol{\sigma} + \mathbf{I} \tag{47}$$

Regular similarity

- Evaluating the new formula iteratively with $\sigma^0 = 0$

$$\sigma^1 = \mathbf{I} \tag{48}$$
$$\sigma^2 = \alpha\mathbf{A} + \mathbf{I} \tag{49}$$
$$\sigma^3 = \alpha^2\mathbf{A}^2 + \alpha\mathbf{A} + \mathbf{I} \tag{50}$$

Regular similarity

- In the limit of a large number of iterations:

$$\sigma = \sum_{m=0}^{\infty} (\alpha \mathbf{A})^m \tag{51}$$

- And also, by rearranging:

$$\sigma = (\mathbf{I} - \alpha \mathbf{A})^{-1} \tag{52}$$

$$\sum_{m=0}^{\infty} (\alpha \mathbf{A})^m = (\mathbf{I} - \alpha \mathbf{A})^{-1} \tag{53}$$

Regular similarity

- This similarity measure includes counts of paths of all lengths
- A weighted count of all the paths between nodes i and j with paths of length r getting weight α^r
- As long as $\alpha < 1$ longer paths get less weight than shorter ones
- In effect we say that two nodes are similar either if they are connected by few short paths or by many long paths

Regular similarity

- The matrix $(\mathbf{I} - \alpha\mathbf{A})$ does not have inverse when $det(\mathbf{I} - \alpha\mathbf{A}) = 0$

$$
\begin{aligned}
det(\mathbf{I} - \alpha\mathbf{A}) &= det(-\alpha(\mathbf{A} - \frac{1}{\alpha}\mathbf{I})) && (54)\\
&= (-\alpha)^n det(\mathbf{A} - \frac{1}{\alpha}\mathbf{I}) && (55)\\
& && (56)
\end{aligned}
$$

- Since $\alpha \neq 0$, there is no inverse when $det(\mathbf{A} - \frac{1}{\alpha}\mathbf{I}) = 0$
- This is characteristic polynomial and the solutions $\frac{1}{\alpha} = \kappa$ are eigenvalues

Regular similarity

- Thus, the matrix does not have inverse (divergence) whenever $\alpha = \frac{1}{\kappa}$
- If we start with small α values and increase it the first time we hit the divergence is when $\alpha = \frac{1}{\kappa_1}$
- After that it happens always when we hit another eigenvalue
- Thus, if we pick $\alpha < \frac{1}{\kappa_1}$ we guarantee convergence

Regular similarity

- "Katz similarity"
- It is a generalization of the structural similarity
- With structural similarity we count common neighbors
- The number of common neighbors is the number of paths of length two
- Our "Katz similarity" counts paths of all lengths and weight them differently

Regular similarity

- Similarly to discussion of PageRank and Katz centrality we can remove the effect of forwarding to much similarity to neighbors by dividing with node degree

$$\sigma_{ij} = \frac{\alpha}{k_i} \sum_k A_{ik} \sigma_{kj} + \delta_{ij} \tag{57}$$

$$\sigma = (\mathbf{D} - \alpha \mathbf{A})^{-1} \mathbf{D} \tag{58}$$

Homophily

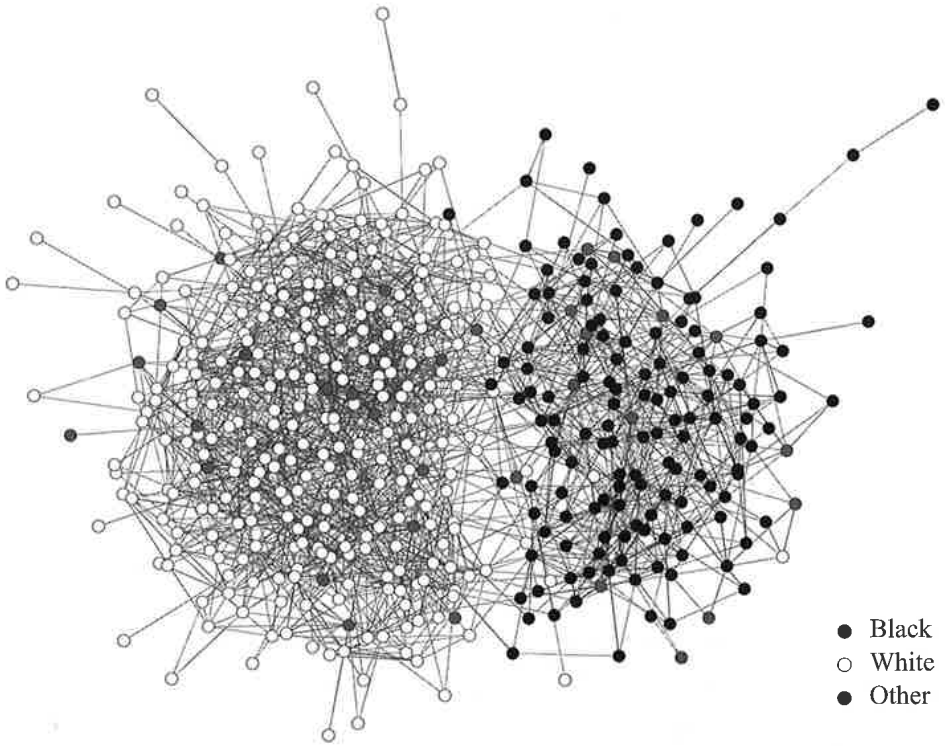


Figure: Friendship network at a US high school

Homophily and assortative mixing

- Division of the network into two groups
- Along lines of a membership in a class, e.g. race
- In social networks this phenomenon has been long observed
- Sociologists have observed such division along side many different dimensions

Homophily and assortative mixing

- People tend to make friends based on all sorts of characteristics, e.g. age, nationality, language, income, etc
- People tend to associate with other who are similar to them
- This tendency is called *homophily* or *assortative mixing*
- Sometimes, *disassortative mixing* is also observed
- This is tendency for people to associate with others who are unlike them, e.g. gender in romantic partnerships

Assortative mixing

- Assortative mixing by enumerative characteristics
- E.g. belonging to a certain class such as nationality, race, or gender
- These are discrete values and in most cases binary values
- Assortative mixing by scalar characteristics
- E.g. age, income, degree, etc.

Assortative mixing by enumerative characteristics

- We have a network in which the nodes are classified according to some characteristic that has a finite set of possible values
- For instance, nodes are people classified by nationality, or gender
- Nodes are Web pages classified by language
- Nodes are Wikipedia pages classified by topic

Assortative mixing by enumerative characteristics

- The network is assortative if a significant fraction of links run between nodes of the same type
- An elegant way to measure the assortiveness is to find the fraction of links that run between nodes of the same type and then subtract the fraction of such links that we would expect in a random network
- If the fraction of links between nodes of the same type equals the expected number then our measure gives 0
- Only if the fraction of links between nodes of the same type is significantly higher than the expected number we will have positive difference

Assortative mixing by enumerative characteristics

- In mathematical terms, let us denote by c_i the class of node i
- Then the total number of links that run between nodes of the same type is:

$$\sum_{links(i,j)} \delta(c_i, cj) = \frac{1}{2} \sum_{ij} A_{ij} \delta(c_i, c_j) \tag{59}$$

Assortative mixing by enumerative characteristics

- What is the expected number of links between nodes i and j
- Let nodes i and j have degrees k_i and k_j respectively
- Now we let j attach the second end of its single link to a random node
- The probability that j selects node i is equal to $\frac{k_i}{2m}$
- In total: the expected number of links between i and j is $\frac{k_j k_i}{2m}$, and the expected number of links between nodes of the same type:

$$\frac{1}{2}\sum_{ij}\frac{k_i k_j}{2m}\delta(c_i, c_j) \tag{60}$$

Assortative mixing by enumerative characteristics

- Taking the difference we get:

$$\frac{1}{2}\sum_{ij} A_{ij}\delta(c_i,c_j) - \frac{1}{2}\sum_{ij}\frac{k_i k_j}{2m}\delta(c_i,c_j) = \frac{1}{2}\sum_{ij}(A_{ij} - \frac{k_i k_j}{2m})\delta(c_i,c_j) \quad (61)$$

- Typically, we will calculate the fraction of such links:

$$Q = \frac{1}{2m}\sum_{ij}(A_{ij} - \frac{k_i k_j}{2m})\delta(c_i,c_j) \quad (62)$$

Assortative mixing by enumerative characteristics

- The quantity Q is called the modularity and is a measure of the extent to which like is connected to like in a network
- It is strictly less than 1
- It takes positive values if there are more links between nodes of the same type then what we would expect by chance
- It takes negative values otherwise
- We can also define $B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$ as an element of matrix $\mathbf{B}$
- Modularity matrix

Assortative mixing by scalar characteristics

- Scalar characteristics allows us to say that two nodes are approximately the same
- E.g. two people are approximately of the same age
- In fact, people tend to associate with others on the basis of such approximate ages
- Thus, if nodes tend to be connected more often with other nodes having a similar characteristic then we say that the network is assortitavely mixed by that characteristic

Assortative mixing by scalar characteristics

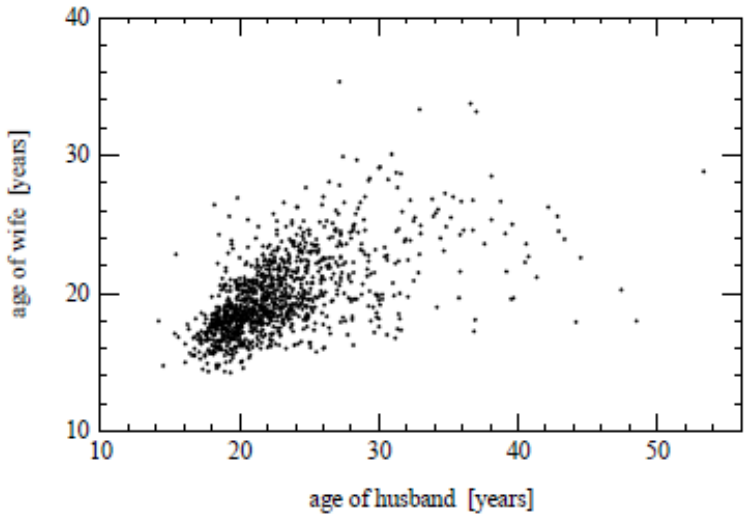


Figure: Assortative mixing by age (from Mixing patterns in networks by Newman)

Assortative mixing by scalar characteristics

- How to measure the magnitude of the assortative mixing
- Let us use again covariance of a scalar quantity over all links
- We have the pairs of values (x_i, x_j) for nodes linked by link (i, j)

Definition

Average μ of the value x_i at the end of a link

$$\mu = \frac{\sum_{ij} A_{ij} x_i}{\sum_{ij} A_{ij}} = \frac{\sum_i k_i x_i}{\sum_i k_i} = \frac{1}{2m} \sum_i k_i x_i \tag{63}$$

Assortative mixing by scalar characteristics

Definition

Covariance of x_i and x_j over links

$$\begin{aligned} \mathrm{cov}(x_i, x_j) &= \frac{\sum_{ij} A_{ij}(x_i - \mu)(x_j - \mu)}{\sum_{ij} A_{ij}} \\ &= \dots \\ &= \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) x_i x_j \end{aligned} \tag{64}$$

Assortative mixing by scalar characteristics

- The covariance is positive if values x_i and x_j et both ends of a link tend to be either both small or both large
- It will be negative if they vary in opposite directions
- Thus, if we have assortative mixing the covariance is positive
- If we have disassortative mixing the covariance is negative

Assortative mixing by scalar characteristics

- We can normalize to obtain 1 for a perfect mixed network
- In a perfectly mixed network x_i and x_j at both ends of a link are always equal
- We put $x_j = x_i$ in the previous equation and obtain the maximal covariance as our normalization constant
- In fact, it is the variance in this case

Assortative mixing by scalar characteristics

Definition

Assortativity coefficient

$$r = \frac{\sum_{ij} (A_{ij} - \frac{k_i k_j}{2m}) x_i x_j}{\sum_{ij} (k_i \delta_{ij} - \frac{k_i k_j}{2m}) x_i x_j} \tag{65}$$

- $-1 \leq r \leq 1$
- $r = 0$ it the values on both ends of links are uncorrelated
- For the data from the previous figure $r = 0.574$

Assortative mixing by degree

- It is of special interest because degree is a network property
- E.g. if we have assortative mixing by degree high-degree nodes tend to connect to other high-degree nodes
- Low-degree nodes tend to connect to other low-degree nodes
- Typically, we obtain a network structure with a *core* of high-degree nodes and a *periphery* of low-degree nodes
- In a disassortative mixing network we obtain a star-like structure where high-degree nodes connect to low-degree nodes

Assortative mixing by degree

Definition

Covariance of x_i and x_j over links

$$\text{cov}(k_i, k_j) = \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) k_i k_j \tag{66}$$

Definition

Assortativity coefficient

$$r = \frac{\sum_{ij} (A_{ij} - \frac{k_i k_j}{2m}) k_i k_j}{\sum_{ij} (k_i \delta_{ij} - \frac{k_i k_j}{2m}) k_i k_j} \tag{67}$$

Assortative mixing by degree

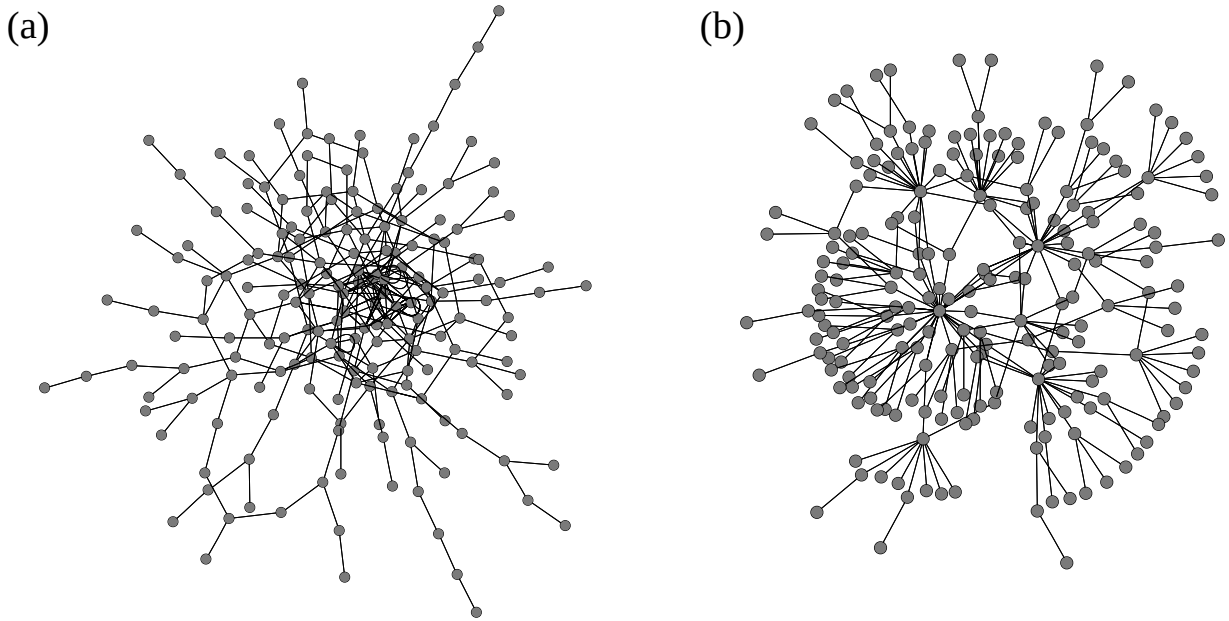


Figure: Assortative mixing by degree (from Mixing patterns and community structure in networks by Newman)

Measures Notebook

- Jupyter Notebook example
- http://kti.tugraz.at/staff/denis/courses/netsci/measures.zip