1. **Alternative relaxation method for modularity maximization and equivalence with spectral clustering:** For modularity maximization we have the network division vector $s$ defined as:

$$s_i = \begin{cases} 
1 & \text{if node } i \text{ belongs to } g_1 \\
-1 & \text{if node } i \text{ belongs to } g_2 
\end{cases}$$

In terms of this vector and the modularity matrix $B$ we express the modularity as:

$$Q = \frac{1}{4m} s^T Rs$$

We have the following two constraints on the division vector:

(a) $s$ always points to one of the $2^n$ corners of a $n$-dimensional hypercube centered on the origin,

(b) $s$ always has the same length, which is $\sqrt{n}$.

Let us now relax these constraints:

(a) We relax the direction constraint and allow $s$ to point in any direction.

(b) We relax the length constraint by: $\sum i k s_i^2 = 2m$. Note that the length of the vector $s$ is constrained by the bounding hyperellipsoid (in the figure: the blue circle is the standard relaxation with the bounding hypersphere, and the green ellipse is the alternative relaxation method with the bounding hyperellipsoid).

Your task is to show that:

(a) the solution of the maximization problem simplifies to (a general eigenvector equation):

$$As = \lambdaDs.$$  

(b) the solution of minimization of the normalized cut size given by $L_n r = \mu r$, where $L_n = D^{-1/2}LD^{-1/2}$ is the normalized Graph Laplacian can be also simplified to $As = \lambda Ds$. Hence, two methods for community detection are in fact equivalent to each other.
2. Give some intuitions and ideas of how we can use random walk (surfer) to extract communities from networks.